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Finite Plane Deformation of Thin Elastic Sheets Reinforced with Inextensible Cords

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FINITE PLANE DEFORMATION OF THIN ELASTIC SHEETS REINFORCED WITH INEXTENSIBLE CORDS

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A general theory is formulated, in tensor notation, for the finite elastic deformation of curvilinearly aeolotropic materials which are subject to constraints, the aeolotropy and the system of constraints being related to different curvilinear frames of reference in the undeformed body.

The results obtained are utilized in considering the general two-dimensional deformation of a thin, plane, uniform sheet of elastic material, reinforced by means of a continuous layer of thin, flexible, inextensible cords lying in the plane midway between its major surfaces. These cords coincide with either or both of the families of curves which constitute a general curvilinear co-ordinate system in the plane in which they lie. The theory is formulated initially for materials which exhibit a type of aeolotropy which is general, except for the restriction required for the structure of the plate to be symmetrical about the reinforcing layer.

When there are two sets of cords, the governing differential equations are hyperbolic, with the characteristic curves coinciding with the paths followed by the cords, and the solution of these equations, subject to various types of boundary condition, is investigated. When the cords lie initially in two sets of parallel straight lines, the stress resultants and displacements can be expressed in terms of real arbitrary functions. This type of solution is employed to examine the deformation of an infinite plane sector, with a system of stresses and displacements prescribed along its edges.

The general theory for a sheet reinforced with a single set of cords is also developed briefly. The linear equations for classically small deformations are derived both for this case, and for a sheet containing two sets of cords, and when the cords lie initially in parallel straight lines these equations are solved in terms of arbitrary functions.

1. Introduction

In work on continuum mechanics, it is often convenient to introduce several systems of co-ordinates to deal with various aspects of the problem under consideration. Such methods have been employed, for example, by Oldroyd (1950) in considering general rheological properties, and by Green & Zerna (1954) in work on finite elastic deformations, and have the advantage that, by a suitable formulation, the transformation of formulae from one system of co-ordinates to another can be achieved by following the ordinary rules of the tensor calculus. A general discussion of the use of various frames of reference in continuum mechanics has been given by Truesdell (1952).

Usually, these additional co-ordinate systems serve to specify the configuration of the continuum in various states of deformation, or at different times, but similar means may be employed to describe the mechanical properties of the body when these are non-isotropic. Thus, to examine materials exhibiting curvilinear aeolotropy, a suitable curvilinear co-ordinate system may be introduced related to points in the undeformed body (Adkins 1955 *a*, *b*). Additional non-isotropic properties may occur owing to the presence of constraints upon the deformation, and materials containing such features have been discussed in general terms by Ericksen & Rivlin (1954). In the present paper, separate co-ordinate systems are introduced to describe aeolotropic properties, systems of constraints and the distribution of stress.

Constraints may occur owing to the presence in an elastic body of thin inextensible cords, and a number of simple problems involving such materials have been solved by Adkins & Rivlin (1955), and by Adkins (1956). In these, it is assumed that the cords are ideally thin, inextensible and perfectly flexible, and that the cords of any one set are sufficiently close together for them to be regarded as a continuous layer in the undeformed body. In the present paper, the general two-dimensional deformation of a thin plane sheet of reinforced elastic material is considered, using the same assumptions regarding the sets of cords in the reinforcing layer. It is assumed also that the sheet is of uniform thickness, and that the cords lie in a plane midway between its major surfaces. An approximation to such conditions may occur, for example, during the stretching of rubberized fabrics or reinforced rubber sheets.

The theory is developed along lines analogous to those followed by Adkins, Green & Nicholas (1954) in considering unreinforced plates, but the general theory of materials subject to constraints has been employed to derive expressions for the stress resultants, it being assumed initially that the reinforcing layer contains two sets of cords following the co-ordinate lines of a suitably chosen curvilinear co-ordinate system. For completeness, a brief discussion has been given for materials exhibiting a suitable type of curvilinear aeolotropy, but in the subsequent work attention is confined mainly to isotropic plates.

When there are two sets of cords, the constraint conditions furnish two equations which involve only the two geometrical parameters defining the deformation in the middle plane of the sheet, and under suitable conditions these may be solved independently of the stress equations. The equations of equilibrium are satisfied by the introduction of an Airy stress function ϕ, yielding the usual relations between this quantity and the stress components. The latter are functions of the deformation which depend upon the nature of the elastic material, and contain, in addition, two parameters introduced by the presence of the constraints. When these parameters are eliminated, there remains a single equation involving ϕ and the geometrical quantities defining the deformation. An additional relation, which serves to determine the variation in thickness throughout the deformed sheet, is obtained for incompressible bodies from the incompressibility condition, and for compressible materials from the condition that the major surfaces are free from applied stress.

It is shown from general considerations that the governing equations for a sheet reinforced with two sets of cords are hyperbolic in character with the characteristic curves coinciding with the paths followed by the cords. In § 8 the solution, subject to various types of boundary condition, is discussed for the case where the cords coincide with a general system of co-ordinates in the middle plane of the undeformed sheet, but attention is subsequently concentrated on the special case where the cords lie initially in parallel straight lines. The stress resultants and the quantities defining the deformation can then be expressed in terms of real arbitrary functions, and the results are similar in character to those obtained by Rivlin (1955) for a network of inextensible cords without elastic material. In § 11 the theory is applied to examine the deformation of an infinite plane sector of reinforced material subject to a specified distribution of displacements and applied forces. In § 12 the linear equations for classically small deformations of a reinforced sheet are derived.

In the final section of the paper, the general theory is developed briefly for a thin plane sheet containing a single set of cords in the reinforcing layer. In this case there is only one constraint condition but two relations involving ϕ. The solution of the linear equations for small deformations then involves functions of both real and complex variables, and is thus intermediate in character between the complex variable forms of ordinary classical elasticity (see, for example, Green & Zerna 1954), and the real variable solution obtained when there are two sets of cords.

2. Preliminary considerations

The notation of the paper is based upon that developed in recent work on finite elastic deformations by Green and his co-workers (see, for example, Green & Zerna 1954), and by Adkins (1955 *a*, *b*).

The points of an unstressed and unstrained body at time $t = 0$ are defined by a system of rectangular Cartesian co-ordinates x^i or by a general curvilinear system of co-ordinates θ^i; the latter move with the body as it is deformed and form a curvilinear system in the strained body at time t. The points of the deformed body are also defined by a system of rectangular co-ordinates y^i, and in the present paper we take the x^i-axes and y^i-axes to coincide. The covariant and contravariant metric tensors for the co-ordinate system θ^i in the unstrained body are denoted by g_{ij}, g^{ij} respectively, and for the co-ordinates θ^i in the strained body at time t, the corresponding metric tensors are G_{ij}, G^{ij}. We write

$$g = |g_{ij}|, \quad G = |G_{ij}|, \tag{2·1}$$

where latin indices take the values 1, 2, 3. Referred to the co-ordinates θ^i, covariant and mixed strain tensors γ_{ij}, γ^i_j may be defined by

$$\left.\begin{aligned}\gamma_{ij} &= \tfrac{1}{2}(G_{ij}-g_{ij}),\\ \gamma^i_j &= g^{ik}\gamma_{kj} = \tfrac{1}{2}(g^{ik}G_{kj}-\delta^i_j),\end{aligned}\right\} \tag{2·2}$$

where δ^i_j is the Kronecker delta.

For isotropic materials the strain-energy function W, measured per unit volume of the unstrained body, can be expressed as a function of three strain invariants I_1, I_2, I_3, which may be written

$$\left.\begin{aligned}I_1 &= 3+2\gamma^r_r = g^{ij}G_{ij},\\ I_2 &= 3+4\gamma^r_r+2(\gamma^r_r\gamma^s_s-\gamma^r_s\gamma^s_r) = I_3 g_{ij}G^{ij},\\ I_3 &= |\delta^r_s+2\gamma^r_s| = G/g.\end{aligned}\right\} \tag{2·3}$$

For aeolotropic materials possessing no symmetry properties, W is a general function of all of the strain components, and the contravariant stress tensor τ^{ij} referred to the curvilinear co-ordinate system θ^i is given by

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}}\left(\frac{\partial W}{\partial\gamma_{ij}}+\frac{\partial W}{\partial\gamma_{ji}}\right). \tag{2·4}$$

When the body is incompressible, $I_3 = 1$, and the stress tensor becomes

$$\tau^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial\gamma_{ij}}+\frac{\partial W}{\partial\gamma_{ji}}\right)+pG^{ij}, \tag{2·5}$$

where p is a scalar function of the co-ordinates θ^i.

Non-isotropic properties of the elastic material may be defined by introducing additional curvilinear co-ordinate systems $\bar{\theta}^i$, θ'^i, ..., related to points in the undeformed body. Metric and strain tensors may be associated with these co-ordinate systems in a manner exactly analogous to the corresponding quantities defined for the co-ordinate system θ^i, and are obtained from the latter in the usual manner by tensor transformations. For example, using the appropriate symbols to distinguish quantities associated with the systems $\bar{\theta}^i$, θ'^i, ..., we have

$$\bar{g}_{ij} = \frac{\partial\theta^r}{\partial\bar{\theta}^i}\frac{\partial\theta^s}{\partial\bar{\theta}^j}g_{rs}, \quad \gamma'^{i}_{j} = \frac{\partial\theta'^i}{\partial\theta^r}\frac{\partial\theta^s}{\partial\theta'^j}\gamma^r_s. \tag{2·6}$$

Elastic aeolotropy and constraints in the elastic material represent obvious properties which may be thus described, although conceivably other features could be imagined for which a similar formulation is possible. We here consider elastic bodies subject to constraints which may be described by means of functional relationships of the form

$$f_m(\bar{\gamma}_{ij}) = 0 \quad (m = 1, 2, \ldots, n) \tag{2·7}$$

between the strain components $\bar{\gamma}_{ij}$ referred to the co-ordinate system $\bar{\theta}^i$. Materials of a similar type have been discussed by Ericksen & Rivlin (1954), who have pointed out that for an elastic deformation to be possible, less than six such constraints must be imposed. The stress-strain relations for such bodies may be obtained from virtual work considerations. Thus, following the usual procedure (see, for example, Green & Zerna 1954), we may establish the virtual-work equation

$$\delta W = \frac{\partial W}{\partial\gamma_{ij}}\delta\gamma_{ij} = \sqrt{\left(\frac{G}{g}\right)}\tau^{ij}\delta\gamma_{ij}, \tag{2·8}$$

where δW is the variation in the strain-energy function at any point arising from permissible variations $\delta\gamma_{ij}$, of the symmetric strain tensor γ_{ij}. For compressible materials not subject to constraints, the stress-strain relations (2·4) follow immediately. If there are constraints of the form (2·7) we must have

$$\frac{\partial f_m}{\partial\bar{\gamma}_{rs}}\delta\bar{\gamma}_{rs} = 0 \quad (m = 1, 2, \ldots, n;\ n<6),$$

or, using transformations of the form (2·6)

$$\frac{\partial f_m}{\partial\bar{\gamma}_{rs}}\frac{\partial\theta^i}{\partial\bar{\theta}^r}\frac{\partial\theta^j}{\partial\bar{\theta}^s}\delta\gamma_{ij} = 0 \quad (m = 1, 2, \ldots, n;\ n<6). \tag{2·9}$$

Since the variations $\delta\gamma_{ij}$ must be subject to the conditions (2·9), and $\delta\gamma_{ij} = \delta\gamma_{ji}$, it follows that

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}}\left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}\right) + \frac{1}{2}\sum_{m=1}^{n} q_m \frac{\partial f_m}{\partial \bar{\gamma}_{rs}}\left(\frac{\partial \theta^i}{\partial \bar{\theta}^r}\frac{\partial \theta^j}{\partial \bar{\theta}^s} + \frac{\partial \theta^i}{\partial \bar{\theta}^s}\frac{\partial \theta^j}{\partial \bar{\theta}^r}\right) \quad (n<6), \tag{2·10}$$

where the quantities q_m are scalar functions of the co-ordinates θ^i. When incompressibility is included among the constraint conditions, $I_3 = 1$, and equations (2·10) are replaced by

$$\tau^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \gamma_{ij}} + \frac{\partial W}{\partial \gamma_{ji}}\right) + pG^{ij} + \frac{1}{2}\sum_{m=1}^{n} q_m \frac{\partial f_m}{\partial \bar{\gamma}_{rs}}\left(\frac{\partial \theta^i}{\partial \bar{\theta}^r}\frac{\partial \theta^j}{\partial \bar{\theta}^s} + \frac{\partial \theta^i}{\partial \bar{\theta}^s}\frac{\partial \theta^j}{\partial \bar{\theta}^r}\right) \quad (n<5). \tag{2·11}$$

The elastic properties of materials exhibiting curvilinear aeolotropy may be described by the introduction of a third system of curvilinear co-ordinates θ'^i related to points in the undeformed body. If this system is suitably chosen, the form of W is the same for all points of the elastic body when expressed as a function of the quantities $\gamma'_{(ij)}$ defined by

$$\gamma'_{(ij)} = \gamma'^i_j \sqrt{(g'_{ii}/g'_{jj})}, \quad \gamma'^i_j = \tfrac{1}{2}(g'^{ik}G'_{kj} - \delta^i_j). \tag{2·12}$$

The terms $\partial W/\partial\gamma_{ij} + \partial W/\partial\gamma_{ji}$ in (2·10) and (2·11) may then be replaced by expressions involving the derivatives $\partial W/\partial\gamma'_{(ij)}$. Thus equations (2·10) become

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}}\sum_{r=1}^{3}\sum_{s=1}^{3}(A^{ij}_{(rs)} + A^{ji}_{(rs)})\frac{\partial W}{\partial \gamma'_{(rs)}} + \frac{1}{2}\sum_{m=1}^{n} q_m \frac{\partial f_m}{\partial \bar{\gamma}_{rs}}\left(\frac{\partial \theta^i}{\partial \bar{\theta}^r}\frac{\partial \theta^j}{\partial \bar{\theta}^s} + \frac{\partial \theta^i}{\partial \bar{\theta}^s}\frac{\partial \theta^j}{\partial \bar{\theta}^r}\right) \quad (n<6), \tag{2·13}$$

where

$$A^{ij}_{(rs)} = \frac{\partial \theta^i}{\partial \theta'^k}\frac{\partial \theta^j}{\partial \theta'^s} g'^{rk}\sqrt{\frac{g'_{rr}}{g'_{ss}}} \quad (r,\ s \text{ not summed}), \tag{2·14}$$

and (2·11) may be similarly modified. Symmetries in the elastic material may restrict the form of the strain-energy function as a function of the strain components, and give rise to further modifications in the form of the terms involving W. For example, if W can be expressed in the form

$$W = W(\Psi_t), \tag{2·15}$$

where

$$\Psi_t = \Psi_t(\gamma'_{(ij)}) \quad (t = 1, 2, \ldots, p) \tag{2·16}$$

are given functions of the strain components $\gamma'_{(ij)}$, (2·13) may be modified to yield

$$\tau^{ij} = \frac{1}{2\sqrt{I_3}}\sum_{t=1}^{p}\left\{\sum_{r=1}^{3}\sum_{s=1}^{3}(A^{ij}_{(rs)} + A^{ji}_{(rs)})\frac{\partial \Psi_t}{\partial \gamma'_{(rs)}}\right\}\frac{\partial W}{\partial \Psi_t} + \frac{1}{2}\sum_{m=1}^{n} q_m \frac{\partial f_m}{\partial \bar{\gamma}_{rs}}\left(\frac{\partial \theta^i}{\partial \bar{\theta}^r}\frac{\partial \theta^j}{\partial \bar{\theta}^s} + \frac{\partial \theta^i}{\partial \bar{\theta}^s}\frac{\partial \theta^j}{\partial \bar{\theta}^r}\right) \quad (n<6). \tag{2·17}$$

Special cases may be discussed along the lines followed by Adkins (1955*a*, 1956), or by Ericksen & Rivlin (1954) for rectilinearly aeolotropic bodies, the latter being regarded as a special case obtained by the choice $\theta'^i = x^i$ for the co-ordinate system defining the aeolotropy.

The stress-strain relations for isotropic bodies may also be obtained as a special case of (2·17) by choosing for the functions Ψ_t the strain invariants I_1, I_2, I_3. For compressible materials we have

$$\tau^{ij} = \frac{2}{\sqrt{I_3}}\left(g^{ij}\frac{\partial W}{\partial I_1} + B^{ij}\frac{\partial W}{\partial I_2} + p'G^{ij}\right) + \frac{1}{2}\sum_{m=1}^{n} q_m \frac{\partial f_m}{\partial \bar{\gamma}_{rs}}\left(\frac{\partial \theta^i}{\partial \bar{\theta}^r}\frac{\partial \theta^j}{\partial \bar{\theta}^s} + \frac{\partial \theta^i}{\partial \bar{\theta}^s}\frac{\partial \theta^j}{\partial \bar{\theta}^r}\right) \quad (n<6), \tag{2·18}$$

where

$$B^{ij} = g^{ij}I_1 - g^{ir}g^{js}G_{rs}, \quad p' = I_3\frac{\partial W}{\partial I_3}. \tag{2·19}$$

The corresponding formulae for incompressible bodies are obtained by putting $I_3 = 1$ and $p' = p$, where p is the scalar function defined for (2·5), and imposing the restriction $n < 5$.

Formally, from the stress-strain relations for a material subject to constraints, the corresponding formulae for one in which some of the constraints are absent may be obtained merely by omitting the unwanted terms. Thus the results (2·4) and (2·5) for compressible and incompressible materials not subject to constraints of the form (2·7), may be obtained from (2·10) and (2·11) respectively by omitting the terms involving f_m, and a similar procedure may be applied to (2·13), (2·17) and (2·18). It must be remembered, however, that the existence of a particular constraint, in general, implies a restriction upon the form of W which may be employed.

If $\mathbf{t}$ is the stress vector associated with a surface in the deformed body whose unit normal $\mathbf{u}$ is given by

$$\mathbf{u} = u_i \mathbf{G}^i, \tag{2·20}$$

then

$$\mathbf{t} = \frac{u_i \mathbf{T}_i}{\sqrt{G}} = u_i \tau^{ij} \mathbf{G}_j = \sum_i u_i \mathbf{t}_i \sqrt{G^{ii}}, \tag{2·21}$$

where

$$\mathbf{T}_i = \sqrt{(GG^{ii})}\, \mathbf{t}_i = \sqrt{(G)}\, \tau^{ij} \mathbf{G}_j \quad (i \text{ not summed}), \tag{2·22}$$

$\mathbf{G}_j$, $\mathbf{G}^j$ are the covariant and contravariant base vectors in the deformed body, and $\mathbf{t}_i$ is the stress vector associated with the surface $\theta^i =$ constant.

In the absence of body forces the equations of equilibrium may now be written

$$\mathbf{T}_{i,i} = 0 \quad \text{or} \quad \tau^{ij} \|_i = 0, \tag{2·23}$$

where the comma denotes partial differentiation with respect to θ^i, and the double line signifies covariant differentiation with respect to the deformed body, that is, with respect to θ^i and the metric tensor components G_{ij}, G^{ij}.

SHEET REINFORCED WITH TWO SETS OF CORDS: GENERAL THEORY

3. The deformation and constraint conditions

Let the undeformed body consist of a thin plate of highly elastic material, bounded by the plane surfaces $x^3 = \pm h_0$, and containing in the middle plane $x^3 = 0$ a layer of thin, perfectly flexible, inextensible cords. This reinforcing layer contains two independent sets of cords lying along the co-ordinate curves of a curvilinear co-ordinate system $\bar{\theta}^\alpha$ in the plane $x^3 = 0$. The cords of each set are assumed to lie sufficiently close together for any plane deformation to be restricted by the condition that there is no extension along any of the $\bar{\theta}^\alpha$ co-ordinate curves in the undeformed body.

It is assumed that the plate is subjected to a finite deformation, symmetrical about the plane $x^3 = 0$, which thus becomes the centre plane $y^3 = 0$ of the deformed body. The major surfaces of the plate after deformation are given by $y^3 = \pm h$, where h is in general a function of y^1, y^2. We choose the moving curvilinear co-ordinate system θ^i so that

$$y^\alpha = y^\alpha(\theta^1, \theta^2, t), \quad y^3 = \theta^3, \tag{3·1}$$

where here, and subsequently, Greek indices take the values 1, 2. If the plate is sufficiently thin we may write approximately

$$x^\alpha = x^\alpha(\theta^1, \theta^2), \quad x^3 = y^3/\lambda = \theta^3/\lambda, \tag{3·2}$$

where λ is a scalar invariant function of θ^1, θ^2.

From (3·1) we have for the metric tensors G_{ij}, G^{ij}

$$\left.\begin{aligned} G_{\alpha\beta} &= A_{\alpha\beta}, \quad G_{\alpha 3} = 0, \quad G_{33} = 1, \\ G^{\alpha\beta} &= A^{\alpha\beta}, \quad G^{\alpha 3} = 0, \quad G^{33} = 1, \end{aligned}\right\} \tag{3·3}$$

with
$$A = |A_{\alpha\beta}|, \quad A^{\alpha\rho}A_{\rho\beta} = \delta^{\alpha}_{\beta}, \tag{3·4}$$

where $A_{\alpha\beta}$, $A^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with co-ordinates θ^α in the middle plane $y^3 = 0$ of the deformed plate.

Similarly, from (3·2), if the plate is thin, we have the approximate forms

$$\left.\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta}, \quad g_{33} = 1/\lambda^2, \\ g^{\alpha\beta} &= a^{\alpha\beta}, \quad g^{33} = \lambda^2, \\ g &= a/\lambda^2, \quad a = |a_{\alpha\beta}|, \end{aligned}\right\} \tag{3·5}$$

where $a_{\alpha\beta}$, $a^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with curvilinear co-ordinates θ^α in the plane $x^3 = 0$ of the undeformed body.

The curves in the undeformed plate followed by the cords may be defined by relations of the form
$$\bar{\theta}^\alpha(x^1, x^2) = \text{constant}, \quad x^3 = 0, \tag{3·6}$$

and when the plate is deformed, these are carried into the curves

$$\bar{\theta}^\alpha\{x^1(y^1, y^2), x^2(y^1, y^2)\} = \text{constant} \tag{3·7}$$

in the plane $y^3 = 0$ of the deformed body. We may therefore associate with the co-ordinate curves $\bar{\theta}^\alpha$, before and after deformation, metric tensors analogous to those defined for the moving co-ordinate system θ^α. In conformity with (3·3) to (3·5) we thus write

$$\left.\begin{aligned} \bar{g}_{\alpha\beta} &= \bar{a}_{\alpha\beta}, \quad \bar{g}^{\alpha\beta} = \bar{a}^{\alpha\beta}, \quad \bar{a} = |\bar{a}_{\alpha\beta}|, \\ G_{\alpha\beta} &= \bar{A}_{\alpha\beta}, \quad \bar{G}^{\alpha\beta} = \bar{A}^{\alpha\beta}, \quad \bar{A} = |\bar{A}_{\alpha\beta}|. \end{aligned}\right\} \tag{3·8}$$

The strain invariants are now given, from (2·3), by

$$\left.\begin{aligned} I_1 &= \lambda^2 + a^{\alpha\beta}A_{\alpha\beta} = \lambda^2 + \bar{a}^{\alpha\beta}\bar{A}_{\alpha\beta}, \\ I_2 &= (A/a)\,(\lambda^2 a_{\alpha\beta}A^{\alpha\beta}+1) = (\bar{A}/\bar{a})\,(\lambda^2\bar{a}_{\alpha\beta}\bar{A}^{\alpha\beta}+1), \\ I_3 &= \lambda^2 A/a = \lambda^2\bar{A}/\bar{a}, \end{aligned}\right\} \tag{3·9}$$

approximately.

The restriction upon the deformation (3·2) imposed by the layer of cords follows readily from geometrical considerations. Thus if corresponding elements of length in the middle plane of the plate before and after deformation are denoted by $\mathrm{d}s_0$, $\mathrm{d}s$ respectively, we have

$$\mathrm{d}s_0^2 = a_{\alpha\beta}\,\mathrm{d}\theta^\alpha\,\mathrm{d}\theta^\beta = \bar{a}_{\alpha\beta}\,\mathrm{d}\bar{\theta}^\alpha\,\mathrm{d}\bar{\theta}^\beta,$$
$$\mathrm{d}s^2 = A_{\alpha\beta}\,\mathrm{d}\theta^\alpha\,\mathrm{d}\theta^\beta = \bar{A}_{\alpha\beta}\,\mathrm{d}\bar{\theta}^\alpha\,\mathrm{d}\bar{\theta}^\beta,$$

and from these expressions, since $\mathrm{d}s_0 = \mathrm{d}s$ along the curves $\bar{\theta}^\alpha = \text{constant}$, we obtain

$$\bar{a}_{11} = \bar{A}_{11}, \quad \bar{a}_{22} = \bar{A}_{22}, \tag{3·10}$$

or
$$\bar{\gamma}_{11} = \bar{\gamma}_{22} = 0. \tag{3·11}$$

Equations (3·10) may evidently be written

$$\frac{\partial x^\alpha}{\partial\bar{\theta}^1}\frac{\partial x^\alpha}{\partial\bar{\theta}^1} = \frac{\partial y^\alpha}{\partial\bar{\theta}^1}\frac{\partial y^\alpha}{\partial\bar{\theta}^1}, \quad \frac{\partial x^\alpha}{\partial\bar{\theta}^2}\frac{\partial x^\alpha}{\partial\bar{\theta}^2} = \frac{\partial y^\alpha}{\partial\bar{\theta}^2}\frac{\partial y^\alpha}{\partial\bar{\theta}^2}. \tag{3·12}$$

If the paths of the cords in the undeformed body are expressed as functional relationships between $\bar{\theta}^\alpha$ and x^α, equations (3·12) serve to determine the co-ordinates y^α, and hence the displacement components in the middle plane of the deformed sheet, as functions of $\bar{\theta}^\alpha$. Conversely, if the configuration of the system of cords in the deformed body is given, so that y^α are known functions of $\bar{\theta}^\alpha$, equations (3·12) may be used to determine relationships between x^α and $\bar{\theta}^\alpha$. We shall henceforth confine our attention to the case where the paths of the cords in the undeformed body are known.

4. Resultant forces in the sheet

Expressions for the stress resultants at any given point in the reinforced plate may be derived by a limiting process, by supposing the layer of cords to be replaced by a thin layer of elastic material subject to constraints of the form (3·11), and bounded initially by the planes $x^3 = \pm\epsilon_0$ $(\epsilon_0 < h_0)$. In this layer, we may employ the stress-strain relations (2·10) and (2·11), or the forms derived from them by introducing the appropriate form of W, while in the remainder of the plate, the simpler relations, which omit the constraint terms, apply. We thus write

$$\left.\begin{aligned} \tau^{ij} &= \tau_e^{ij} \qquad (\epsilon_0 \leqslant |x^3| \leqslant h_0), \\ \tau^{ij} &= \tau_e'^{ij} + \tau_c^{ij} \quad (0 \leqslant |x^3| \leqslant \epsilon_0), \end{aligned}\right\} \tag{4·1}$$

where τ_e^{ij}, $\tau_e'^{ij}$ are given by formulae of the type (2·4) or (2·5), whilst from (2·10) or (2·11) and (3·11)

$$\tau_c^{ij} = q_1 \frac{\partial\theta^i}{\partial\bar{\theta}^1}\frac{\partial\theta^j}{\partial\bar{\theta}^1} + q_2 \frac{\partial\theta^i}{\partial\bar{\theta}^2}\frac{\partial\theta^j}{\partial\bar{\theta}^2}. \tag{4·2}$$

The existence of constraints implies that the form of W employed in calculating $\tau_e'^{ij}$ is not necessarily identical with that entering into the expression for τ_e^{ij}, but since the term involving $\tau_e'^{ij}$ disappears in the limit, this distinction is unimportant.

In both regions, we assume the elastic material to be uniform in the x^3-direction, and the form of its strain-energy function to be restricted so that there is elastic symmetry with respect to the x^3-plane at each point. We may then, with appropriate modifications, follow the procedure employed by Adkins *et al.* (1954) in considering unreinforced isotropic plates, and replace the forces acting across either of the surfaces $\theta^\alpha =$ constant in the deformed body by a physical stress resultant $\mathbf{n}_\alpha$, measured per unit length of the curve $\theta^\alpha =$ constant in the plane $y^3 = 0$, and given by

$$\mathbf{n}_\alpha \sqrt{A^{\alpha\alpha}} = n^{\alpha i}\mathbf{G}_i \quad (\alpha = 1, 2;\ i = 1, 2, 3), \tag{4·3}$$

where

$$n^{\alpha i} = \int_{-\epsilon}^{\epsilon} (\tau_e'^{\alpha i} + \tau_c^{\alpha i})\,\mathrm{d}y^3 + \int_{\epsilon}^{h} \tau_e^{\alpha i}\,\mathrm{d}y^3 + \int_{-h}^{-\epsilon} \tau_e^{\alpha i}\,\mathrm{d}y^3, \tag{4·4}$$

and $\epsilon = \lambda\epsilon_0$. Since both the unstrained body, and the deformation to which it is subjected, are symmetrical about $x^3 = 0$, $n^{\alpha 3}$ and the stress couples are zero.

Remembering (4·2) and (3·2), the first integral term in the expression (4·4) for $n^{\alpha\beta}$ may be written

$$\int_{-\epsilon}^{\epsilon} \tau_e'^{\alpha\beta}\,\mathrm{d}y^3 + \frac{\partial\theta^\alpha}{\partial\bar{\theta}^1}\frac{\partial\theta^\beta}{\partial\bar{\theta}^1}\int_{-\epsilon}^{\epsilon} q_1\,\mathrm{d}y^3 + \frac{\partial\theta^\alpha}{\partial\bar{\theta}^2}\frac{\partial\theta^\beta}{\partial\bar{\theta}^2}\int_{-\epsilon}^{\epsilon} q_2\,\mathrm{d}y^3. \tag{4·5}$$

If we allow ϵ to become indefinitely small, while q_1, q_2 increase in such a manner that the terms $\int_{-\epsilon}^{\epsilon} q_\alpha \, dy^3$ remain finite, the first integral in (4·5) vanishes, and equations (4·4) yield

$$n^{\alpha\beta} = n_e^{\alpha\beta} + n_c^{\alpha\beta}, \tag{4·6}$$

where

$$\left.\begin{aligned} n_e^{\alpha\beta} &= \int_{-h}^{h} \tau_e^{\alpha\beta} \, dy^3, \\ n_c^{\alpha\beta} &= \bar{\sigma}_1 \frac{\partial\theta^\alpha}{\partial\bar{\theta}^1}\frac{\partial\theta^\beta}{\partial\bar{\theta}^1} + \bar{\sigma}_2 \frac{\partial\theta^\alpha}{\partial\bar{\theta}^2}\frac{\partial\theta^\beta}{\partial\bar{\theta}^2}, \end{aligned}\right\} \tag{4·7}$$

and the quantities $\bar{\sigma}_\alpha$ are arbitrary scalar functions of θ^α. The functions defined by (4·6) and (4·7) are (plane) surface tensors. In the co-ordinate system θ^α, the physical stress resultants are given by

$$n_{(\alpha\beta)} = n^{\alpha\beta} \sqrt{(A_{\beta\beta}/A^{\alpha\alpha})}, \tag{4·8}$$

the brackets indicating that these quantities are not tensors.

For the relations between the stress components at the major surfaces of the deformed plate, the discussion of Adkins *et al.* (1954, § 3), which depends only upon the existence of symmetry about $y^3 = 0$, applies here without modification. If, therefore, the surfaces $y^3 = \pm h$ are free from applied forces, we may write down the relations

$$[\tau_e^{\alpha3} - \tau_e^{\alpha\beta} y^3_{,\beta}]_{-h}^{h} = 0, \quad [\tau_e^{33} - \tau_e^{3\beta} y^3_{,\beta}]_{-h}^{h} = 0, \tag{4·9}$$

from which by eliminating $\tau_e^{\alpha3}$ we obtain

$$\tau_e^{33} - \tau_e^{\alpha\beta} y^3_{,\alpha} y^3_{,\beta} = 0 \tag{4·10}$$

at $y^3 = \pm h$, a comma denoting differentiation with respect to θ^α. Thus, if the plate is sufficiently thin, and if $h_{,\alpha}$ is of the same order of magnitude as h, τ_e^{33} is small compared with $\tau_e^{\alpha\beta}$ at the surface of the plate. We therefore assume τ_e^{33} to be negligible throughout the plate and write

$$\tau^{33} = \tau_e^{33} = 0. \tag{4·11}$$

The results of the present section evidently apply for any plate with structural and elastic symmetry about the centre plane $x^3 = 0$.

5. Tensions in the cords

Formulae for the stress resultants in the composite sheet may also be obtained by calculating separately, from statical considerations, the forces in the deformed layer of cords, and adding to them the stress resultants for the unreinforced elastic material. Relations of the type (4·6) and (4·7) are again obtained, but in place of the scalars $\bar{\sigma}_\alpha$, we have expressions involving the tensions in the individual cords and the geometrical parameters of the reinforcing layer. To express $\bar{\sigma}_\alpha$ in terms of these quantities, it will be sufficient to calculate the forces in the layer of cords referred to the rectangular Cartesian co-ordinate system y^α.

At any given point x^α in the undeformed sheet, let δ_1 be the distance between adjacent cords of the set lying along the curves $\bar{\theta}^2 =$ constant, this distance being measured along the curve $\bar{\theta}^1 =$ constant through that point, and let δ_2 be similarly defined for the other set of cords. We may then regard $1/\delta_\alpha$ as a measure of the density of the cords following the

$\bar{\theta}^\alpha$-curves at any point. After deformation, each of these cords carries a tension τ_α, and we write $\sigma_\alpha = \tau_\alpha/\delta_\alpha$. It is not necessary to assume that δ_α are continuous functions of $\bar{\theta}^\alpha$, provided that these quantities are at no point too large for the assumptions of § 3 to be justified. It follows from subsequent work, however, that for a continuous distribution of stress in the reinforcing layer, the quotients σ_α must be continuous.

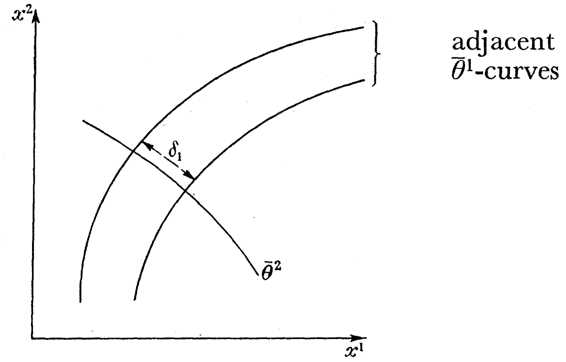

FIGURE 1*a*

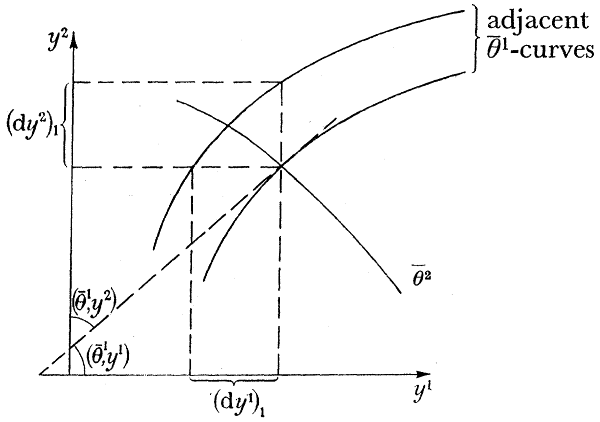

FIGURE 1*b*

Referred to the rectangular Cartesian co-ordinates y^α in the middle plane $y^3 = 0$ of the deformed sheet, the formulae for $n_c^{\alpha\beta}$ may now be written

$$\left.\begin{aligned}
n_c^{11} &= \frac{\tau_1}{(\mathrm{d}y^2)_1}\cos(\bar{\theta}^1, y^1) + \frac{\tau_2}{(\mathrm{d}y^2)_2}\cos(\bar{\theta}^2, y^1), \\
n_c^{12} &= \frac{\tau_1}{(\mathrm{d}y^2)_1}\cos(\bar{\theta}^1, y^2) + \frac{\tau_2}{(\mathrm{d}y^2)_2}\cos(\bar{\theta}^2, y^2), \\
n_c^{21} &= -\left\{\frac{\tau_1}{(\mathrm{d}y^1)_1}\cos(\bar{\theta}^1, y^1) + \frac{\tau_2}{(\mathrm{d}y^1)_2}\cos(\bar{\theta}^2, y^1)\right\}, \\
n_c^{22} &= -\left\{\frac{\tau_1}{(\mathrm{d}y^1)_1}\cos(\bar{\theta}^1, y^2) + \frac{\tau_2}{(\mathrm{d}y^1)_2}\cos(\bar{\theta}^2, y^2)\right\},
\end{aligned}\right\} \qquad (5{\cdot}1)$$

where $(\mathrm{d}y^\alpha)_\beta$ is the distance between the intersections of adjacent $\bar{\theta}^\beta$-curves with a line parallel to the y^α-axis, and $(\bar{\theta}^\beta, y^\alpha)$ is the angle between the tangent to one of these $\bar{\theta}^\beta$-curves and the y^α-axis in the deformed body. The situation before and after deformation is illustrated by figures 1 *a* and 1 *b* respectively.

The signs prefixing the quantities occurring in (5·1) are chosen so that the contribution to $n_c^{\alpha\alpha}$ of a $\bar{\theta}^\beta$-cord has the same sign as τ_β, and is therefore positive if the cord is in tension. To determine these signs, we observe that the direction cosines of the tangent to the curve $\bar{\theta}^2 = C$ in the deformed layer are

$$\left.\begin{aligned} \cos(\bar{\theta}^1, y^1) &= \frac{\partial\bar{\theta}^2}{\partial y^2}\Big/\left\{\left(\frac{\partial\bar{\theta}^2}{\partial y^1}\right)^2 + \left(\frac{\partial\bar{\theta}^2}{\partial y^2}\right)^2\right\}^{\frac{1}{2}}, \\ \cos(\bar{\theta}^1, y^2) &= -\frac{\partial\bar{\theta}^2}{\partial y^1}\Big/\left\{\left(\frac{\partial\bar{\theta}^2}{\partial y^1}\right)^2 + \left(\frac{\partial\bar{\theta}^2}{\partial y^2}\right)^2\right\}^{\frac{1}{2}}, \end{aligned}\right\} \tag{5·2}$$

and these are positive if the tangent lies in the first quadrant, as shown in figure 2. In this case, if τ_1 is positive, the contributions of this cord to the stresses n_c^{12}, n_c^{21} are also positive, and from the first or second of (5·1), $(\mathrm{d}y^2)_1$ must therefore be measured in the positive sense. Moreover, since in passing to a neighbouring curve the increments of y^α and C are connected by the relation

$$\frac{\partial\bar{\theta}^2}{\partial y^\alpha}\mathrm{d}y^\alpha = \mathrm{d}C, \tag{5·3}$$

we may infer, by considering a positive increment dC and remembering (5·2), that $(\mathrm{d}y^2)_1$ and $(\mathrm{d}y^1)_1$ are measured in the positive and negative senses respectively. The negative sign must therefore be associated with $(\mathrm{d}y^1)_1$ in (5·1). The other cases which arise may be considered similarly.

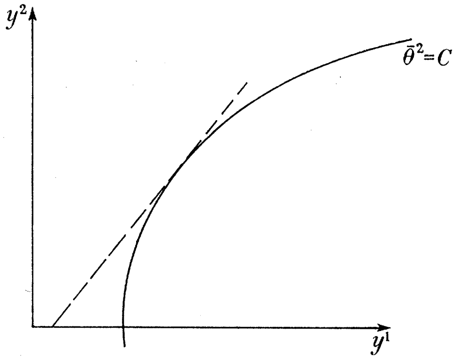

FIGURE 2

To express $(\mathrm{d}y^\alpha)_\beta$ in terms of the distances δ_α we consider the curves $\bar{\theta}^\alpha$ = constant in the undeformed sheet. In passing from $\bar{\theta}^2(x^1, x^2) = k$ to a neighbouring curve, we have as before

$$\frac{\partial\bar{\theta}^2}{\partial x^\alpha}\mathrm{d}x^\alpha = \mathrm{d}k, \tag{5·4}$$

and with this must be coupled the relations

$$\mathrm{d}y^\alpha = \frac{\partial y^\alpha}{\partial x^\beta}\mathrm{d}x^\beta. \tag{5·5}$$

By putting dy^2, dy^1 successively equal to zero, and eliminating dx^α from (5·4) and (5·5) we obtain

$$(\mathrm{d}y^1)_1 = \left\{\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)}\Big/\frac{\partial(\bar{\theta}^2, y^2)}{\partial(x^1, x^2)}\right\}\mathrm{d}k, \quad (\mathrm{d}y^2)_1 = -\left\{\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)}\Big/\frac{\partial(\bar{\theta}^2, y^1)}{\partial(x^1, x^2)}\right\}\mathrm{d}k. \tag{5·6}$$

If the displacement dk is measured along a $\bar{\theta}^2$-curve, we have

$$\mathrm{d}\bar{\theta}^1 = \frac{\partial\bar{\theta}^1}{\partial x^\alpha}\,\mathrm{d}x^\alpha = 0, \tag{5·7}$$

and δ_1 is then obtained by introducing the values of dx^1, dx^2 derived from (5·4) and (5·7) into

$$\delta_1 = \{(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\}^{\frac{1}{2}}, \tag{5·8}$$

giving

$$\delta_1 = \left\{\left[\left(\frac{\partial\bar{\theta}^1}{\partial x^1}\right)^2 + \left(\frac{\partial\bar{\theta}^1}{\partial x^2}\right)^2\right]^{\frac{1}{2}} \Big/ \frac{\partial(\bar{\theta}^1,\bar{\theta}^2)}{\partial(x^1,x^2)}\right\}\mathrm{d}k. \tag{5·9}$$

Also, from relations of the form

$$\frac{\partial x^\alpha}{\partial\bar{\theta}^\lambda}\frac{\partial\bar{\theta}^\lambda}{\partial x^\beta} = \delta^\alpha_\beta,$$

we may obtain

$$\left(\frac{\partial\bar{\theta}^1}{\partial x^1}, \frac{\partial\bar{\theta}^1}{\partial x^2}, \frac{\partial\bar{\theta}^2}{\partial x^1}, \frac{\partial\bar{\theta}^2}{\partial x^2}\right) = \left(\frac{\partial x^2}{\partial\bar{\theta}^2}, -\frac{\partial x^1}{\partial\bar{\theta}^2}, -\frac{\partial x^2}{\partial\bar{\theta}^1}, \frac{\partial x^1}{\partial\bar{\theta}^1}\right)\frac{\partial(\bar{\theta}^1,\bar{\theta}^2)}{\partial(x^1,x^2)}, \tag{5·10}$$

and hence

$$\frac{\partial(\bar{\theta}^2,y^\alpha)}{\partial(x^1,x^2)} = -\left\{\frac{\partial x^2}{\partial\bar{\theta}^1}\frac{\partial y^\alpha}{\partial x^2} + \frac{\partial x^1}{\partial\bar{\theta}^1}\frac{\partial y^\alpha}{\partial x^1}\right\}\frac{\partial(\bar{\theta}^1,\bar{\theta}^2)}{\partial(x^1,x^2)} = -\frac{\partial y^\alpha}{\partial\bar{\theta}^1}\frac{\partial(\bar{\theta}^1,\bar{\theta}^2)}{\partial(x^1,x^2)}. \tag{5·11}$$

By introducing the value of dk obtained from (5·9) into (5·6), we obtain

$$(\mathrm{d}y^1)_1 = -\left\{\frac{\partial(y^1,y^2)}{\partial(\bar{\theta}^1,\bar{\theta}^2)}\Big/\left(\frac{\partial y^2}{\partial\bar{\theta}^1}\sqrt{\bar{a}_{22}}\right)\right\}\delta_1, \quad (\mathrm{d}y^2)_1 = \left\{\frac{\partial(y^1,y^2)}{\partial(\bar{\theta}^1,\bar{\theta}^2)}\Big/\left(\frac{\partial y^1}{\partial\bar{\theta}^1}\sqrt{\bar{a}_{22}}\right)\right\}\delta_1. \tag{5·12}$$

Similarly, by using relations of the type (5·10) with x^α replaced by y^α throughout, the formulae (5·2) may be reduced to

$$\cos(\bar{\theta}^1, y^\alpha) = \frac{\partial y^\alpha}{\partial\bar{\theta}^1}\Big/\sqrt{\bar{a}_{11}}. \tag{5·13}$$

Results analogous to (5·12) and (5·13), but with $\bar{\theta}^1$, $\bar{\theta}^2$ interchanged, hold for $(\mathrm{d}y^\alpha)_2$, $\cos(\bar{\theta}^2, y^\alpha)$, respectively, and since

$$\frac{\partial(y^1,y^2)}{\partial(\bar{\theta}^1,\bar{\theta}^2)} = \sqrt{\bar{A}},$$

these, with (5·1), yield

$$n_c^{\alpha\beta} = \left\{\sigma_1\sqrt{\left(\frac{\bar{a}_{22}}{\bar{a}_{11}}\right)}\frac{\partial y^\alpha}{\partial\bar{\theta}^1}\frac{\partial y^\beta}{\partial\bar{\theta}^1} + \sigma_2\sqrt{\left(\frac{\bar{a}_{11}}{\bar{a}_{22}}\right)}\frac{\partial y^\alpha}{\partial\bar{\theta}^2}\frac{\partial y^\beta}{\partial\bar{\theta}^2}\right\}\Big/\sqrt{\bar{A}}. \tag{5·14}$$

By putting $\theta^\alpha = y^\alpha$ in (4·7) and comparing with (5·14) we obtain finally

$$\bar{\sigma}_1 = \sigma_1\sqrt{\left(\frac{\bar{a}_{22}}{\bar{a}_{11}\bar{A}}\right)}, \quad \bar{\sigma}_2 = \sigma_2\sqrt{\left(\frac{\bar{a}_{11}}{\bar{a}_{22}\bar{A}}\right)}. \tag{5·15}$$

The same results could have been obtained by a direct calculation of the formulae for $n_c^{\alpha\beta}$ in the general system of co-ordinates θ^α, but the equations to be handled are then more cumbersome.

6. STRESSES IN THE ELASTIC MATERIAL

The elastic contribution to the stress resultants $n^{\alpha\beta}$ may be evaluated by using the appropriate form of the stress-strain formulae given in § 2. We consider first, materials which exhibit a suitable type of curvilinear aeolotropy defined by choosing the co-ordinate system θ'^i so that

$$\theta'^\alpha = \theta'^\alpha(x^1, x^2), \quad \theta'^3 = x^3. \tag{6·1}$$

The assumption of § 4 that the elastic material is symmetrical at each point about the plane parallel to the major surfaces of the undeformed sheet, requires that the form of W, regarded as a function of the quantities $\gamma'_{(ij)}$, remains unchanged under the transformation of co-ordinates

$$(\theta'^1, \theta'^2, \theta'^3) = (\theta^{*1}, \theta^{*2}, -\theta^{*3}).$$

Remembering (2·12), this implies that W must be restricted to the form

$$W = W(\gamma'_{(\alpha\beta)}, \gamma'_{(33)}, \gamma'_{(\alpha3)}\gamma'_{(3\beta)}). \tag{6·2}$$

By using (3·1), (3·2) and (6·1) we may now obtain relations analogous to (3·3), (3·4) and (3·5) in which the metric tensor components $G_{\alpha\beta}, \ldots, A_{\alpha\beta}, \ldots$ are replaced by the corresponding primed quantities $G'_{\alpha\beta}, \ldots, A'_{\alpha\beta}, \ldots$, associated with the co-ordinates θ'^i. From these, with (2·12), we may derive approximate expressions for the strain components $\gamma'_{(ij)}$. In the absence of constraint conditions, the formulae (2·13) or (2·17) then yield, for compressible materials, the approximate formulae

$$\left.\begin{aligned} \tau_e^{\alpha\beta} &= \frac{1}{2\sqrt{I_3}} \sum_{\lambda=1}^{2} \sum_{\mu=1}^{2} \{A^{\alpha\beta}_{(\lambda\mu)} + A^{\beta\alpha}_{(\lambda\mu)}\} \frac{\partial W}{\partial \gamma'_{(\lambda\mu)}}, \\ \tau_e^{33} &= \frac{\lambda^2}{\sqrt{I_3}} \frac{\partial W}{\partial \gamma'_{(33)}}, \end{aligned}\right\} \tag{6·3}$$

where
$$A^{\alpha\beta}_{(\lambda\mu)} = \frac{\partial\theta^\alpha}{\partial\theta'^\gamma}\frac{\partial\theta^\beta}{\partial\theta'^\mu} a'^{\lambda\gamma} \sqrt{\frac{a'_{\lambda\lambda}}{a'_{\mu\mu}}} \quad (\lambda, \mu \text{ not summed}). \tag{6·4}$$

For incompressible materials we have

$$\left.\begin{aligned} \tau_e^{\alpha\beta} &= \frac{1}{2} \sum_{\lambda=1}^{2} \sum_{\mu=1}^{2} \{A^{\alpha\beta}_{(\lambda\mu)} + A^{\beta\alpha}_{(\lambda\mu)}\} \frac{\partial W}{\partial \gamma'_{(\lambda\mu)}} + pA^{\alpha\beta}, \\ \tau_e^{33} &= \lambda^2 \frac{\partial W}{\partial \gamma'_{(33)}} + p, \end{aligned}\right\} \tag{6·5}$$

or, if we make use of (4·11),

$$\tau_e^{\alpha\beta} = \frac{1}{2} \sum_{\lambda=1}^{2} \sum_{\mu=1}^{2} \{A^{\alpha\beta}_{(\lambda\mu)} + A^{\beta\alpha}_{(\lambda\mu)}\} \frac{\partial W}{\partial \gamma'_{(\lambda\mu)}} - \lambda^2 A^{\alpha\beta} \frac{\partial W}{\partial \gamma'_{(33)}}. \tag{6·6}$$

Bodies containing symmetries in the elastic material additional to that defined by (6·2) may be similarly discussed, and the passage to the more usual case of rectilinear aeolotropy may be made in each case by allowing the co-ordinate systems θ'^i and x^i to coincide.

The corresponding formulae for the isotropic case have been given by Adkins *et al.* (1954), and are obtained from (2·18), with the additional constraint terms omitted. For incompressible materials, in place of (6·6) we have

$$\tau_e^{\alpha\beta} = 2\left\{\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right) a^{\alpha\beta} + \left[\left(\frac{A}{a} + \lambda^4 - \lambda^2 I_1\right)\frac{\partial W}{\partial I_2} - \lambda^2 \frac{\partial W}{\partial I_1}\right] A^{\alpha\beta}\right\}. \tag{6·7}$$

In the compressible case it is often more convenient to replace I_1, I_2, I_3 by three different, mutually independent invariants J_1, J_2, J_3 given by

$$\left.\begin{aligned} J_1 &= I_1 - 3, \\ J_2 &= I_2 - 2I_1 + 3, \\ J_3 &= I_3 - I_2 + I_1 - 1. \end{aligned}\right\} \tag{6·8}$$

The formulae for $\tau_e^{\alpha\beta}$ then become

$$\tau_e^{\alpha\beta} = \frac{2}{\surd I_3}\left\{\left[\frac{\partial W}{\partial J_1} + (\lambda^2-2)\frac{\partial W}{\partial J_2} - (\lambda^2-1)\frac{\partial W}{\partial J_3}\right]a^{\alpha\beta} + \frac{I_3}{\lambda^2}\left[\frac{\partial W}{\partial J_2} + (\lambda^2-1)\frac{\partial W}{\partial J_3}\right]A^{\alpha\beta}\right\}, \qquad (6\cdot 9)$$

whilst the condition $\tau_e^{33} = 0$ yields

$$\lambda^2\frac{\partial W}{\partial J_1} + \lambda^2(J_1+1-\lambda^2)\frac{\partial W}{\partial J_2} + \{(1-\lambda^2)(J_1+1-\lambda^2)+J_2+J_3\}\frac{\partial W}{\partial J_3} = 0. \qquad (6\cdot 10)$$

From (3·1), (3·2) and (6·1), and the consequent formulae for the metric tensors of the type (3·3) to (3·5), it follows that the strain components $\gamma'_{(ij)}$, the invariants I_r, J_r and hence also W and $\tau_e^{\alpha\beta}$, are, to a sufficient degree of approximation, independent of y^3 (or θ^3). From (4·7) we then obtain

$$n_e^{\alpha\beta} = 2\lambda h_0 \tau_e^{\alpha\beta}, \qquad (6\cdot 11)$$

where $\tau_e^{\alpha\beta}$ are given by the formulae of the present section appropriate to the material under consideration.

7. EQUATIONS OF EQUILIBRIUM: AIRY'S STRESS FUNCTION

In this section the theory is developed along lines similar to those followed by Adkins *et al.* (1954) in the treatment of unreinforced plates. Thus, if the major surfaces of the deformed sheet are free from applied forces, the equations of equilibrium may be written

$$n^{\alpha\beta}\,\|_{\alpha} = 0, \qquad (7\cdot 1)$$

where the double line now denotes covariant differentiation with respect to the plane variables θ^α in the deformed body using the Christoffel symbols formed from the metric tensors $A_{\alpha\beta}$, $A^{\alpha\beta}$. These equations may be obtained by integrating (2·23) through the thickness of the sheet and making use of surface conditions of the type (4·9), or by a direct consideration of the stress resultants in the plane $y^3 = 0$.

Equations (7·1) may be satisfied by introducing a scalar invariant function ϕ of the co-ordinates θ^α, which is Airy's stress function for the deformed body, and is such that

$$\phi\,\|_{\alpha\beta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\rho}n^{\gamma\rho}, \qquad (7\cdot 2)$$

or

$$n^{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\phi\,\|_{\gamma\rho} = (a/A)\,({}_0\epsilon^{\alpha\gamma})\,({}_0\epsilon^{\beta\rho})\,\phi\,\|_{\gamma\rho}, \qquad (7\cdot 3)$$

where

$$\left.\begin{aligned} \epsilon^{\alpha\beta}\surd A = \epsilon_{\alpha\beta}/\surd A = {}_0\epsilon^{\alpha\beta}\surd a = {}_0\epsilon_{\alpha\beta}/\surd a &= 1 \quad \text{if} \quad \alpha = 1,\ \beta = 2,\\ &= -1 \quad \text{if} \quad \alpha = 2,\ \beta = 1,\\ &= 0 \quad \text{otherwise},\end{aligned}\right\} \qquad (7\cdot 4)$$

and

$$\epsilon_{\alpha\rho}\epsilon^{\alpha\lambda} = \delta^\lambda_\rho.$$

From (4·6), (4·7) and (7·3) we may now write

$$n^{\alpha\beta} = n_e^{\alpha\beta} + \sum_{\nu=1}^{2}\bar{\sigma}_\nu\frac{\partial\theta^\alpha}{\partial\bar{\theta}^\nu}\frac{\partial\theta^\beta}{\partial\bar{\theta}^\nu} = \epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\phi\,\|_{\gamma\rho}, \qquad (7\cdot 5)$$

and hence

$$\left.\begin{aligned}\frac{\partial\bar{\theta}^\lambda}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^\mu}{\partial\theta^\beta}n^{\alpha\beta} &= \frac{\partial\bar{\theta}^\lambda}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^\mu}{\partial\theta^\beta}n_e^{\alpha\beta} + \sum_{\nu=1}^{2}\delta^\lambda_\nu\delta^\mu_\nu\bar{\sigma}_\nu\\ &= \epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\frac{\partial\bar{\theta}^\lambda}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^\mu}{\partial\theta^\beta}\phi\,\|_{\gamma\rho}.\end{aligned}\right\} \qquad (7\cdot 6)$$

When $\lambda = \mu = \nu$ we have

$$\left.\begin{aligned}\bar{\sigma}_\nu &= \frac{\partial\bar{\theta}^\nu}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^\nu}{\partial\theta^\beta}(n^{\alpha\beta}-n_e^{\alpha\beta})\\ &= \frac{\partial\bar{\theta}^\nu}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^\nu}{\partial\theta^\beta}(\epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\phi\,\|_{\gamma\rho}-n_e^{\alpha\beta}) \quad (\nu \text{ not summed}),\end{aligned}\right\} \tag{7·7}$$

whilst the choice $\lambda = 1$, $\mu = 2$ yields

$$\frac{\partial\bar{\theta}^1}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^2}{\partial\theta^\beta}n^{\alpha\beta} = \frac{\partial\bar{\theta}^1}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^2}{\partial\theta^\beta}n_e^{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\frac{\partial\bar{\theta}^1}{\partial\theta^\alpha}\frac{\partial\bar{\theta}^2}{\partial\theta^\beta}\phi\,\|_{\gamma\rho}. \tag{7·8}$$

If we choose the moving curvilinear co-ordinate system θ^α to coincide with the system $\bar{\theta}^\alpha$ defining the directions of the cords, (7·7) and (7·8) reduce to

$$\bar{\sigma}_\nu = \bar{n}^{\nu\nu}-\bar{n}_e^{\nu\nu} = \bar{\epsilon}^{\nu\rho}\bar{\epsilon}^{\nu\rho}\phi\,\|_{\rho\rho}-\bar{n}_e^{\nu\nu} \quad (\nu \text{ not summed}), \tag{7·9}$$

$$\bar{n}^{12} = \bar{n}_e^{12} = -\phi\,\|_{12}/\bar{A}, \tag{7·10}$$

where the additional bars are used to indicate quantities defined with respect to the $\bar{\theta}^\alpha$-curves in the deformed body, and the double bar now denotes covariant differentiation with respect to the co-ordinates $\bar{\theta}^\alpha$ and the metric tensors $\bar{A}_{\alpha\beta}$, $\bar{A}^{\alpha\beta}$.

If we regard the deformation in the middle plane $y^3 = 0$ of the deformed sheet to be determined by a specification of the quantities y^α as functions of x^α or $\bar{\theta}^\alpha$, it follows from the equations of §§ 2 and 6 that the components of strain, and hence also the strain-energy function W, the stresses $\tau_e^{\alpha\beta}$, and the stress-resultant components $n_e^{\alpha\beta}$, may be regarded as functions of y^α and λ. For compressible materials, (7·8) or (7·10), together with the constraint conditions (3·12) and the relation (4·11), therefore furnish four equations for the determination of the four unknowns y^α, λ and ϕ. For incompressible materials the incompressibility condition $I_3 = 1$ takes the place of (4·11). The remaining stress resultants $n^{\alpha\alpha}$ are obtained from (7·3), and the tensions τ_α in the cords from (7·7) or (7·9) and (5·15).

SOLUTION OF THE EQUATIONS

8. The general case

If the co-ordinates y^α are regarded as the dependent variables in (3·12), these equations may be written

$$y^2_{,\alpha} = \{\bar{a}_{\alpha\alpha}-(y^1_{,\alpha})^2\}^{\frac{1}{2}}, \tag{8·1}$$

where a comma now denotes differentiation with respect to $\bar{\theta}^\alpha$, and we choose the sign of the square root so that (8·1) reduces to an identity when $y^\alpha = x^\alpha$. (The subsequent argument would be unaffected by an alternative choice of signs.) By differentiation we may obtain

$$y^2_{,12} = \frac{\bar{a}_{11,2}-2y^1_{,1}y^1_{,12}}{2\{\bar{a}_{11}-(y^1_{,1})^2\}^{\frac{1}{2}}} = \frac{\bar{a}_{22,1}-2y^1_{,2}y^1_{,12}}{2\{\bar{a}_{22}-(y^1_{,2})^2\}^{\frac{1}{2}}},$$

or

$$2\{y^1_{,1}[\bar{a}_{22}-(y^1_{,2})^2]^{\frac{1}{2}}-y^1_{,2}[\bar{a}_{11}-(y^1_{,1})^2]^{\frac{1}{2}}\}y^1_{,12} = \bar{a}_{11,2}[\bar{a}_{22}-(y^1_{,2})^2]^{\frac{1}{2}}-\bar{a}_{22,1}[\bar{a}_{11}-(y^1_{,1})^2]^{\frac{1}{2}}, \tag{8·2}$$

and a similar equation holds for the derivative $y^2_{,12}$ with the figures 1, 2 interchanged throughout.

It follows from (7·10) and (8·2) that the differential equations for y^α and ϕ are hyperbolic, with the characteristic curves coinciding with the paths followed by the cords. Such equations may be discussed along the lines followed by Webster (1933), Hill (1950), Rivlin (1955)† and other workers for similar systems. Thus, if y^1 is specified as a continuous function of position along the parts OA, OB of the characteristics through O (figure 3), and has continuous derivatives of all orders along these lines, its value may be determined at all points of the region bounded by the characteristic curves OA, OB, AC and BC. For we may regard the derivatives $\partial^n y^1/\partial(\bar{\theta}^1)^n$ as known functions of position along the $\bar{\theta}^1$-curve OA, and similarly the derivatives $\partial^n y^1/\partial(\bar{\theta}^2)^n$ are known along the $\bar{\theta}^2$-curve OB. Also, since $y^1_{,1}$ and $y^1_{,2}$ are known at the point O, the value of the second derivative $y^1_{,12}$ may be determined at that point from (8·2). Higher-order derivatives of the type $\partial^{m+n}y^1/\partial(\bar{\theta}^1)^m\,\partial(\bar{\theta}^2)^n$ may be obtained (in principle) by further differentiation of this equation. We may thus, by Taylor's theorem, obtain an expansion for y^1 valid within a finite region around O bounded by the characteristics OA, OB. If PA', QB' are characteristics lying sufficiently close to OA, OB respectively within the region $OACB$, and which intersect at R, y^1 may now be regarded as known along PR, RQ and we may therefore repeat the procedure to obtain expansions for y^1 about the points P, Q. In this way, values for y^1 can be obtained at all points of PA', QB', and the process continued until the whole of the area $OACB$ has been covered. The values of $y^2_{,\alpha}$ may then be found at all points from (8·1), and y^2 thus determined throughout the entire region apart from a constant. The value of this constant may be made definite by fixing the point O.

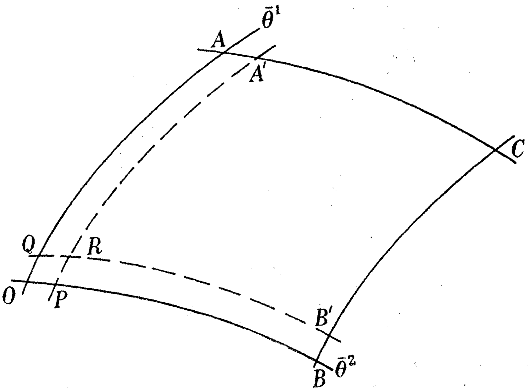

FIGURE 3

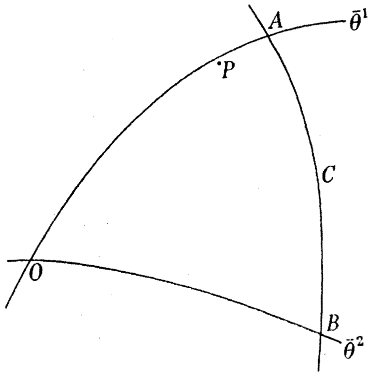

FIGURE 4

Similarly, if the displacement components, and hence y^α, are prescribed along a section AB of a curve C which at no point coincides with a characteristic, the deformation may be determined within the region bounded by AB and the characteristics OA, OB through its ends (figure 4). For if we express the co-ordinates of any point of C in terms of a parameter t,‡ so that $x^\alpha = x^\alpha(t)$ or $\bar{\theta}^\alpha = \bar{\theta}^\alpha(t)$, the quantities y^α and hence their derivatives $\mathrm{d}y^\alpha/\mathrm{d}t$ may be regarded as known functions of t along C. Writing for brevity

$$\frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = \dot{y}^\alpha, \quad \frac{\mathrm{d}\bar{\theta}^\alpha}{\mathrm{d}t} = p_\alpha(t) = p_\alpha, \tag{8·3}$$

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† The author had an opportunity of seeing a copy of Rivlin's paper in the proof stage after a draft of the present paper had been prepared. This led to a number of changes in the present section and stimulated the treatment of §§9 and 10.

‡ This notation is only used in the present section, so that no confusion need arise with the use of t to denote time in §2.

we then have, along C,
$$\dot{y}^\alpha = y^\alpha_{,1} p_1 + y^\alpha_{,2} p_2, \tag{8·4}$$
from which
$$\left.\begin{aligned}(y^1_{,1} p_1)^2 &= (\dot{y}^1)^2 - 2\dot{y}^1 p_2 y^1_{,2} + (y^1_{,2} p_2)^2,\\ (y^2_{,1} p_1)^2 &= (\dot{y}^2)^2 - 2\dot{y}^2 p_2 y^2_{,2} + (y^2_{,2} p_2)^2.\end{aligned}\right\} \tag{8·5}$$
Adding, and making use of (3·12), we have
$$2p_2 \dot{y}^2 y^2_{,2} = \chi - 2p_2 \dot{y}^1 y^1_{,2}, \tag{8·6}$$
where
$$\chi = (\dot{y}^1)^2 + (\dot{y}^2)^2 + p_2^2 \bar{a}_{22} - p_1^2 \bar{a}_{11}. \tag{8·7}$$
By eliminating $y^2_{,2}$ between (8·6) and the second of the constraint conditions (3·12) we obtain
$$4p_2^2\{(\dot{y}^1)^2 + (\dot{y}^2)^2\}(y^1_{,2})^2 - 4p_2 \dot{y}^1 \chi y^1_{,2} + \chi^2 - 4p_2^2 (\dot{y}^2)^2 \bar{a}_{22} = 0,$$
which may be solved to yield
$$y^1_{,2} = \frac{\chi \pm \gamma \Psi}{2p_2(1+\gamma^2)\,\dot{y}^1}, \tag{8·8}$$
where
$$\left.\begin{aligned}\Psi &= \{4p_2^2[(\dot{y}^1)^2 + (\dot{y}^2)^2]\,\bar{a}_{22} - \chi^2\}^{\frac{1}{2}}\\ \text{and}\qquad \gamma &= \dot{y}^2/\dot{y}^1.\end{aligned}\right\} \tag{8·9}$$
For the values of the remaining first derivatives along C, we have from (8·4) and (8·6)
$$y^2_{,2} = \frac{\gamma\chi \mp \Psi}{2p_2(1+\gamma^2)\,\dot{y}^1}, \quad y^\alpha_{,1} = (\dot{y}^\alpha - y^\alpha_{,2} p_2)/p_1. \tag{8·10}$$

The choice of sign occurring in (8·8) and (8·10) may be made definite by introducing the condition that
$$\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)} > 0. \tag{8·11}$$
Thus we have
$$\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)} = \frac{\partial(y^1, y^2)}{\partial(\bar{\theta}^1, \bar{\theta}^2)} \frac{\partial(\bar{\theta}^1, \bar{\theta}^2)}{\partial(x^1, x^2)} = \frac{\partial(y^1, y^2)}{\partial(\bar{\theta}^1, \bar{\theta}^2)} \Big/ \sqrt{\bar{a}}, \tag{8·12}$$
where $\sqrt{\bar{a}}$ is uniquely determined by the equations of the paths of the cords in the undeformed sheet. Also, from (8·8) and (8·10),
$$\frac{\partial(y^1, y^2)}{\partial(\bar{\theta}^1, \bar{\theta}^2)} = y^1_{,1} y^2_{,2} - y^1_{,2} y^2_{,1} = \mp \frac{\Psi}{2p_1 p_2}, \tag{8·13}$$
the upper signs corresponding in each of these equations. Only one of the solutions given by (8·8) and (8·10) can therefore satisfy the inequality (8·11). We may thus obtain unique values of the first derivatives y^α, along C, unless $t = \bar{\theta}^1$ or $t = \bar{\theta}^2$, in which case equations (8·4) are identically satisfied and the method fails.

The second derivatives $y^\alpha_{,\beta\gamma}$ may be obtained by differentiation. For we may put
$$y^\alpha_{,\beta} = A_{\alpha\beta}(t), \tag{8·14}$$
where $A_{\alpha\beta}(t)$ are now known functions of t along C, and thus obtain
$$y^\alpha_{,\beta\gamma} p_\gamma = \frac{\mathrm{d}A_{\alpha\beta}}{\mathrm{d}t} \quad \left(p_\gamma = \frac{\mathrm{d}\bar{\theta}^\gamma}{\mathrm{d}t}\right). \tag{8·15}$$

Also, by differentiation of the constraint conditions (3·12) with respect to $\bar{\theta}^\alpha$, we have

$$y^\gamma_{,\alpha}y^\gamma_{,\alpha\alpha} = \tfrac{1}{2}\bar{a}_{\alpha\alpha,\alpha} \quad (\alpha \text{ not summed}). \tag{8·16}$$

The six equations obtained by giving α, β all possible combinations of values in (8·15) and (8·16) are linear in $y^\alpha_{,\beta\gamma}$ and furnish a solution unless the determinant

$$\Delta = \begin{vmatrix} p_1 & p_2 & 0 & 0 & 0 & 0 \\ 0 & p_1 & p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1 & p_2 & 0 \\ 0 & 0 & 0 & 0 & p_1 & p_2 \\ y^1_{,1} & 0 & 0 & y^2_{,1} & 0 & 0 \\ 0 & 0 & y^1_{,2} & 0 & 0 & y^2_{,2} \end{vmatrix}$$

of their coefficients is zero. This determinant may be reduced to

$$\Delta = p_1^2 p_2^2 (y^1_{,1}y^2_{,2} - y^1_{,2}y^2_{,1}) = \left(\frac{\mathrm{d}\bar{\theta}^1}{\mathrm{d}t}\frac{\mathrm{d}\bar{\theta}^2}{\mathrm{d}t}\right)^2 \sqrt{A}, \tag{8·17}$$

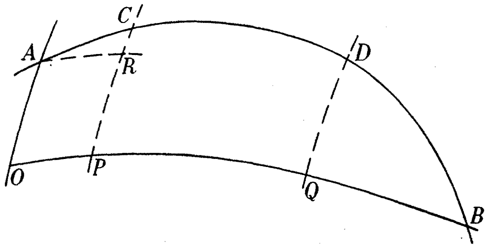

FIGURE 5

which evidently vanishes only if $t = \bar{\theta}^1$ or $t = \bar{\theta}^2$, that is, if C coincides with a characteristic. A similar result is obtained if we replace either or both of the relations (8·16) by

$$y^\gamma_{,\alpha}y^\gamma_{,\alpha\beta} = \tfrac{1}{2}\bar{a}_{\alpha\alpha,\beta} \quad (\beta \neq \alpha;\ \alpha \text{ not summed}). \tag{8·18}$$

By continued differentiation, we may, in principle, determine the third- and higher-order derivatives at all points of C provided $t \neq \bar{\theta}^\alpha$. By Taylor's theorem, the co-ordinates y^α of points in the deformed body may then be expanded as functions of $\bar{\theta}^\alpha$ which are analytic and satisfy (3·12) within a region bounded by C, and which also satisfy the boundary conditions along this curve. This procedure cannot, in general, however, be employed to obtain values of y^α outside the region OAB unless further boundary conditions are prescribed. This may be seen by considering the change in the values of y^α in crossing the $\bar{\theta}^1$-curve OA from a point P just inside the region OAB. From continuity considerations y^α, and hence also the tangential derivatives $y^\alpha_{,1}$ must be continuous in crossing OA, but this is not necessarily true for the normal derivatives, and thus for $y^\alpha_{,2}$. In fact, at points adjoining the curve OA just outside the region OAB, the latter quantities may be assigned any values which satisfy the second of equations (3·12). From the formulae of §§ 4 to 6 we may expect to find that discontinuities in the values of $y^\alpha_{,\beta}$ are accompanied by corresponding changes in $\bar{n}_e^{\alpha\beta}$ and σ_α.

If the values of y^α are given along a curve AB, which coincides with a characteristic along a section CD (figure 5), we may employ Rivlin's method to show that y^α may be determined throughout the region bounded by *AB* and the characteristics *OA*, *OB* through its ends. For since the values of y^α are given along *AC*, *BD* they may be determined at all points within the regions *ARC*, *DQB* bounded by the curve *AB* and the characteristics through *A*, *C*, *D*, *B*. The values of y^α are thus known along the characteristics *CD*, *DQ* and can therefore be determined throughout the region *CDQP*. A similar argument may now be applied to the remaining part *ARPO* of the region *ACDBO*.

If the deformation has been determined, the equation for ϕ may be discussed along similar lines and corresponding results obtained. For example, a specification of two components of applied stress along the curve *C* of figure 4, is sufficient to determine the stress distribution throughout the region *ABO*. For by choosing $\theta^\alpha = y^\alpha$ in (7·2) and (7·8), and writing, for brevity $\partial\overline{\theta}^\alpha/\partial y^\beta = \overline{\theta}^\alpha_{,\beta}$, $\partial^2\phi/\partial y^\alpha\,\partial y^\beta = \phi_{,\alpha\beta}$, we have

$$n^{22} = \phi_{,11}, \quad n^{11} = \phi_{,22}, \quad n^{12} = -\phi_{,12}, \tag{8·19}$$

$$\overline{\theta}^1_{,1}\overline{\theta}^2_{,1}\phi_{,22} - (\overline{\theta}^1_{,1}\overline{\theta}^2_{,2} + \overline{\theta}^1_{,2}\overline{\theta}^2_{,1})\,\phi_{,12} + \overline{\theta}^1_{,2}\overline{\theta}^2_{,2}\phi_{,11} = \overline{\theta}^1_{,\alpha}\overline{\theta}^2_{,\beta}n^{\alpha\beta}_e. \tag{8·20}$$

From (8·19) we see that the boundary conditions imply two relationships between the derivatives $\phi_{,\alpha\beta}$. These, with (8·20), determine values of $\phi_{,\alpha\beta}$ along *C*, provided all three equations are independent. Higher-order derivatives may be found, as before, by differentiation and an expansion for ϕ obtained by Taylor's theorem, throughout the region *ABO*. The process again breaks down if *C* is a characteristic.

9. Solution for straight cords

When the cords lie initially in straight lines in the undeformed body, the metric tensor components $\overline{a}_{\alpha\beta}$ are constants, and the constraint conditions (3·12) may be solved to yield

$$y^1 = f_1(\overline{\theta}^1) + f_2(\overline{\theta}^2), \quad y^2 = g_1(\overline{\theta}^1) + g_2(\overline{\theta}^2), \tag{9·1}$$

where f_1, f_2, g_1 and g_2 are functions which satisfy the equations

$$f_1'^2 + g_1'^2 = \overline{a}_{11}, \quad f_2'^2 + g_2'^2 = \overline{a}_{22}, \tag{9·2}$$

but are otherwise arbitrary.

The metric tensors $\overline{A}_{\alpha\beta}$, $\overline{A}^{\alpha\beta}$ now become

$$\left.\begin{aligned} &\overline{A}_{11} = \overline{a}_{11} = f_1'^2 + g_1'^2, \quad \overline{A}_{22} = \overline{a}_{22} = f_2'^2 + g_2'^2, \\ &\overline{A}_{12} = f_1'f_2' + g_1'g_2', \\ &\sqrt{\overline{A}} = \frac{\partial(y^1, y^2)}{\partial(\overline{\theta}^1, \overline{\theta}^2)} = f_1'g_2' - f_2'g_1', \\ &\overline{A}^{11} = \overline{A}_{22}/\overline{A}, \quad \overline{A}^{22} = \overline{A}_{11}/\overline{A}, \quad \overline{A}^{12} = -\overline{A}_{12}/\overline{A}, \end{aligned}\right\} \tag{9·3}$$

and the Christoffel symbols formed with these components reduce to

$$\left.\begin{aligned} &\Gamma^1_{11} = -\overline{A}_{12}\overline{A}_{12,1}/\overline{A} = (g_2'f_1'' - f_2'g_1'')/\sqrt{\overline{A}}, \\ &\Gamma^2_{22} = -\overline{A}_{12}\overline{A}_{12,2}/\overline{A} = (f_1'g_2'' - g_1'f_2'')/\sqrt{\overline{A}}, \\ &\Gamma^2_{11} = \overline{A}_{11}\overline{A}_{12,1}/\overline{A} = (f_1'g_1'' - g_1'f_1'')/\sqrt{\overline{A}}, \\ &\Gamma^1_{22} = \overline{A}_{22}\overline{A}_{12,2}/\overline{A} = (g_2'f_2'' - f_2'g_2'')/\sqrt{\overline{A}}, \\ &\Gamma^1_{12} = \Gamma^2_{12} = 0, \end{aligned}\right\} \tag{9·4}$$

where, in deriving these results, we have used the relations

$$f_1'f_1''+g_1'g_1''=0,\quad f_2'f_2''+g_2'g_2''=0, \tag{9·5}$$

obtained from (9·2) by differentiation.

From the formulae of §§ 2, 3 and 6, we see that the strain components $\gamma'_{(ij)}$, the invariants I_r, J_r, and hence also the strain-energy function W and the elastic stress resultant components $\bar{n}_e^{\alpha\beta}$, can be expressed in terms of $\bar{a}_{\alpha\beta}$, $\bar{A}_{\alpha\beta}$ and λ. For incompressible materials the condition $I_3=1$ yields

$$\lambda^2=\bar{a}/\bar{A}, \tag{9·6}$$

and for compressible bodies λ may be determined (in principle) in terms of $\bar{a}_{\alpha\beta}$, $\bar{A}_{\alpha\beta}$ from the condition $\tau_e^{33}=0$. We may therefore regard the quantities $\bar{n}_e^{\alpha\beta}$ as known functions of f_1', f_2', g_1' and g_2'. From (9·4) and (7·10) we thus have

$$\phi\,\|_{12}=\phi_{,12}=-\bar{A}\bar{n}_e^{12}, \tag{9·7}$$

a comma now denoting differentiation with respect to $\bar{\theta}^\alpha$. This may be formally integrated to yield

$$\phi=F(\bar{\theta}^1,\bar{\theta}^2)+h_1(\bar{\theta}^1)+h_2(\bar{\theta}^2), \tag{9·8}$$

where $F(\bar{\theta}^1,\bar{\theta}^2)$ is a function chosen so that

$$F_{,12}=-\bar{A}\bar{n}_e^{12}=-(f_1'g_2'-f_2'g_1')^2\,\bar{n}^{12}, \tag{9·9}$$

and h_1 and h_2 are arbitrary functions of their arguments. The remaining stress resultant components are given, from (7·9), (9·4) and (9·8) by

$$\left.\begin{aligned}\bar{n}^{11}&=\phi\,\|_{22}/\bar{A}\\&=\{(F_{,22}+h_2'')\,(f_1'g_2'-f_2'g_1')-(F_{,2}+h_2')\,(f_1'g_2''-g_1'f_2'')\\&\qquad\qquad+(F_{,1}+h_1')\,(f_2'g_2''-g_2'f_2'')\}/(f_1'g_2'-f_2'g_1')^3,\\ \bar{n}^{22}&=\{(F_{,11}+h_1'')\,(f_1'g_2'-f_2'g_1')-(F_{,1}+h_1')\,(g_2'f_1''-f_2'g_1'')\\&\qquad\qquad+(F_{,2}+h_2')\,(g_1'f_1''-f_1'g_1'')\}/(f_1'g_2'-f_2'g_1')^3,\end{aligned}\right\} \tag{9·10}$$

or

$$\left.\begin{aligned}\bar{n}^{11}&=\left\{\frac{\partial}{\partial\bar{\theta}^2}\left(\frac{F_{,2}+h_2'}{\sqrt{\bar{A}}}\right)+\frac{f_2'g_2''-g_2'f_2''}{\bar{A}}\,(F_{,1}+h_1')\right\}\Big/\sqrt{\bar{A}},\\ \bar{n}^{22}&=\left\{\frac{\partial}{\partial\bar{\theta}^1}\left(\frac{F_{,1}+h_1'}{\sqrt{\bar{A}}}\right)+\frac{g_1'f_1''-f_1'g_1''}{\bar{A}}\,(F_{,2}+h_2')\right\}\Big/\sqrt{\bar{A}}.\end{aligned}\right\} \tag{9·11}$$

The tensions τ_α may then be evaluated from the relations

$$\left.\begin{aligned}\tau_1&=\delta_1\sigma_1=\delta_1(\bar{n}^{11}-\bar{n}_e^{11})\sqrt{\left(\frac{\bar{A}\bar{a}_{11}}{\bar{a}_{22}}\right)},\\ \tau_2&=\delta_2\sigma_2=\delta_2(\bar{n}^{22}-\bar{n}_e^{22})\sqrt{\left(\frac{\bar{A}\bar{a}_{22}}{\bar{a}_{11}}\right)},\end{aligned}\right\} \tag{9·12}$$

obtained by combining (5·15) and (7·9).

By omitting the terms involving F, we may derive from (9·10) and (9·12) formulae equivalent to those obtained by Rivlin for a network of inextensible cords.

10. THE FORCE ACROSS A CURVE

An expression for the force across a curve in terms of the arbitrary functions f_α, g_α, h_α, may be derived from the results obtained by Adkins *et al.* (1954). Let $\mathbf{a}_\alpha$, $\mathbf{A}_\alpha$ denote the contravariant base vectors for the moving co-ordinate system θ^α in the middle plane of the undeformed body and of the deformed body respectively. If $\mathbf{v}$ is the displacement vector, we may write

$$\left.\begin{aligned} \mathbf{v} &= v^\alpha \mathbf{a}_\alpha, \\ \text{and} \qquad \mathbf{A}_\alpha &= \mathbf{a}_\alpha + \frac{\partial \mathbf{v}}{\partial \theta^\alpha} = \mathbf{a}_\alpha + v^\beta \,|_\alpha\, \mathbf{a}_\beta, \end{aligned}\right\} \tag{10·1}$$

where the single line denotes covariant differentiation with respect to the undeformed body, that is, with respect to θ^α and the metric tensor components $a_{\alpha\beta}$, $a^{\alpha\beta}$.

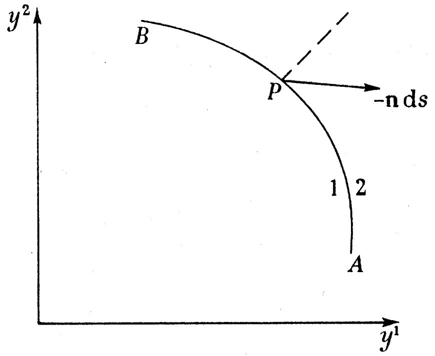

FIGURE 6

We consider any curve AB (figure 6) which lies in the middle plane $y^3 = 0$ of the deformed sheet, and does not intersect itself. If ds denotes an element of the line AB, the total force $\mathbf{P}$ exerted by the region 1 on region 2 across the arc AP is

$$\mathbf{P} = -\int_A^P \mathbf{n}\,\mathrm{d}s = \epsilon^{\rho\beta} \frac{\partial \phi}{\partial \theta^\rho} \mathbf{A}_\beta, \tag{10·2}$$

where

$$\mathbf{n} = \int_{-h}^{h} \mathbf{t}\,\mathrm{d}y^3$$

is the stress resultant per unit length of this curve. From (10·1) and (10·2) we have

$$\begin{aligned} \mathbf{P} &= \epsilon^{\rho\beta} \frac{\partial \phi}{\partial \theta^\rho} (\mathbf{a}_\beta + v^\gamma \,|_\beta\, \mathbf{a}_\gamma) \\ &= \frac{1}{\sqrt{A}} \left\{ -\left[(1 + v^1 \,|_1) \frac{\partial \phi}{\partial \theta^2} - v^1 \,|_2 \frac{\partial \phi}{\partial \theta^1} \right] \mathbf{a}_1 + \left[(1 + v^2 \,|_2) \frac{\partial \phi}{\partial \theta^1} - v^2 \,|_1 \frac{\partial \phi}{\partial \theta^2} \right] \mathbf{a}_2 \right\}. \end{aligned} \tag{10·3}$$

If now we choose the co-ordinate system θ^α so that $\theta^\alpha = x^\alpha$, the base vectors $\mathbf{a}_\alpha$ coincide with the unit vectors $\mathbf{i}_\alpha$ along the x^α-axes. Also

$$v^\alpha = y^\alpha - x^\alpha$$

are now the components of displacement in the co-ordinate system x^α, the covariant derivatives become ordinary derivatives with respect to x^α, and

$$\sqrt{A} = \frac{\partial(y^1, y^2)}{\partial(x^1, x^2)}.$$

Thus, from (10·3) we obtain

$$\mathbf{P} = \left\{-\left(\frac{\partial y^1}{\partial x^1}\frac{\partial \phi}{\partial x^2} - \frac{\partial y^1}{\partial x^2}\frac{\partial \phi}{\partial x^1}\right)\mathbf{i}_1 + \left(\frac{\partial y^2}{\partial x^2}\frac{\partial \phi}{\partial x^1} - \frac{\partial y^2}{\partial x^1}\frac{\partial \phi}{\partial x^2}\right)\mathbf{i}_2\right\}\Big/\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)}$$

$$= -\left\{\frac{\partial(y^1, \phi)}{\partial(\bar{\theta}^1, \bar{\theta}^2)}\mathbf{i}_1 + \frac{\partial(y^2, \phi)}{\partial(\bar{\theta}^1, \bar{\theta}^2)}\mathbf{i}_2\right\}\Big/\frac{\partial(y^1, y^2)}{\partial(\bar{\theta}^1, \bar{\theta}^2)}. \tag{10·4}$$

Writing

$$\mathbf{P} = P_1\mathbf{i}_1 + P_2\mathbf{i}_2, \tag{10·5}$$

and making use of (9·1) and (9·8) we therefore have

$$\left.\begin{aligned} P_1 &= (y^1_{,2}\phi_{,1} - y^1_{,1}\phi_{,2})/(y^1_{,1}y^2_{,2} - y^1_{,2}y^2_{,1}) \\ &= \{f_2'(F_{,1} + h_1') - f_1'(F_{,2} + h_2')\}/(f_1'g_2' - f_2'g_1'), \\ P_2 &= \{g_2'(F_{,1} + h_1') - g_1'(F_{,2} + h_2')\}/(f_1'g_2' - f_2'g_1'), \end{aligned}\right\} \tag{10·6}$$

and hence

$$\left.\begin{aligned} h_1' &= f_1'P_2 - g_1'P_1 - F_{,1}, \\ h_2' &= f_2'P_2 - g_2'P_1 - F_{,2}. \end{aligned}\right\} \tag{10·7}$$

Using formulae analogous to (10·6), Rivlin (1955) has expressed the deformation at any point of a plane network of inextensible cords in terms of the forces acting around its edges. The presence of the elastic terms, represented by $F_{,\alpha}$ appears to exclude this method in the present case.

11. Deformation of an infinite plane sector

From the results of §§ 8 and 10 we may write down the solution to the problem of an infinite plane sector deformed subject to a prescribed system of stresses and displacements along its boundaries. We suppose a known continuous distribution of force to be applied to one edge, which we may choose to coincide with the x^1-axis, while the values of y^α are given along the straight line $ax^1 + bx^2 = 0$. Remembering (10·6), the boundary conditions can then be put in the forms

$$\left.\begin{aligned} P_\alpha &= P_\alpha(x^1) \quad \text{when} \quad x^2 = 0, \\ \frac{\mathrm{d}y^\alpha}{\mathrm{d}t} &= q_\alpha(t) \quad \text{when} \quad x^1 = -bt,\ x^2 = at, \end{aligned}\right\} \tag{11·1}$$

where $P_\alpha(x^1)$ and $q_\alpha(t)$ are known functions of their arguments.

If the cords lie initially along the straight lines

$$\bar{\theta}^1 = x^1 + \alpha x^2 = \text{constant}, \quad \bar{\theta}^2 = \beta x^1 + x^2 = \text{constant}, \tag{11·2}$$

where α and β are constants, we have

$$\bar{a}_{11} = \frac{1+\beta^2}{(1-\alpha\beta)^2}, \quad \bar{a}_{22} = \frac{1+\alpha^2}{(1-\alpha\beta)^2}, \quad \bar{a}_{12} = -\frac{\alpha+\beta}{(1-\alpha\beta)^2}, \quad \bar{a} = \frac{1}{(1-\alpha\beta)^2}, \tag{11·3}$$

and, with the notation of § 8,

$$p_1 = \frac{\mathrm{d}\bar{\theta}^1}{\mathrm{d}t} = a\alpha - b, \quad p_2 = \frac{\mathrm{d}\bar{\theta}^2}{\mathrm{d}t} = a - b\beta. \tag{11·4}$$

From (11·1), (11·3), (11·4) and (8·7) to (8·10) we may therefore evaluate the derivatives $y^{\alpha}_{,\beta}$ in terms of the parameter t along $ax^1+bx^2=0$, and remembering (9·1) write

$$f_1' = A_1(t), \quad f_2' = A_2(t), \quad g_1' = A_3(t), \quad g_2' = A_4(t), \tag{11·5}$$

where $A_r(t)$ are known functions of t. Moreover, along this boundary we have

$$\bar{\theta}^1 = (\alpha a - b)\,t, \quad \bar{\theta}^2 = (a - \beta b)\,t,$$

so that the conditions (11·5) may be satisfied by putting

$$f_1'(\bar{\theta}^1) = A_1\left(\frac{\bar{\theta}^1}{\alpha a - b}\right), \quad f_2'(\bar{\theta}^2) = A_2\left(\frac{\bar{\theta}^2}{a - \beta b}\right), \quad g_1'(\bar{\theta}^1) = A_3\left(\frac{\bar{\theta}^1}{\alpha a - b}\right), \quad g_2'(\bar{\theta}^2) = A_4\left(\frac{\bar{\theta}^2}{a - \beta b}\right). \tag{11·6}$$

Combining (11·1) with (10·7) and making use of (11·6) and (11·2), we have along $x^2=0$

$$\left.\begin{aligned} h_1'(x^1) &= A_1\left(\frac{x^1}{\alpha a - b}\right) P_2(x^1) - A_3\left(\frac{x^1}{\alpha a - b}\right) P_1(x^1) - F_{,1}(x^1, \beta x^1), \\ h_2'(\beta x^1) &= A_2\left(\frac{\beta x^1}{a - \beta b}\right) P_2(x^1) - A_4\left(\frac{\beta x^1}{a - \beta b}\right) P_1(x^1) - F_{,2}(x^1, \beta x^1), \end{aligned}\right\} \tag{11·7}$$

and the forms of h_1', h_2' which satisfy these conditions are evidently obtained by replacing x^1 by $\bar{\theta}^1$ in the first equation and βx^1 by $\bar{\theta}^2$ in the second.

12. The equations for small deformations

For sufficiently small deformations, the relations reduce, as in the classical theory of elasticity, to linear forms. Thus we may put

$$y^{\alpha} = x^{\alpha} + u^{\alpha}, \quad \lambda = 1 + \lambda', \tag{12·1}$$

in the preceding equations, where u^{α} and λ' are small, and neglect squares and products of these quantities and their derivatives. We then have

$$\left.\begin{aligned} &\bar{A}_{\alpha\beta} = \bar{a}_{\alpha\beta} + \bar{a}'_{\alpha\beta}, \\ \text{where} \quad &\bar{a}'_{\alpha\beta} = x^{\mu}_{,\alpha} u^{\mu}_{,\beta} + x^{\mu}_{,\beta} u^{\mu}_{,\alpha}, \end{aligned}\right\} \tag{12·2}$$

$$\left.\begin{aligned} \text{and} \quad &\bar{A} = |\bar{A}_{\alpha\beta}| = \bar{a}(1 + \bar{a}'), \\ \text{where} \quad &\bar{a}' = (\bar{a}_{22}\bar{a}'_{11} + \bar{a}_{11}\bar{a}'_{22} - 2\bar{a}_{12}\bar{a}'_{12})/\bar{a} = -2\bar{a}_{12}\bar{a}'_{12}/\bar{a}. \end{aligned}\right\} \tag{12·3}$$

The constraint conditions (3·12) thus become

$$\bar{a}'_{\alpha\alpha} = 0 \quad \text{or} \quad x^{\mu}_{,\alpha} u^{\mu}_{,\alpha} = 0. \tag{12·4}$$

We confine our attention to isotropic materials, and for these, equations (3·9) and (6·8) for the strain invariants I_r, J_r reduce to

$$I_1 - 3 = \tfrac{1}{2}(I_2 - 3) = I_3 - 1 = J_1 = 2\lambda' + \bar{a}', \tag{12·5}$$

J_2 and J_3 involving only powers of the second and higher order in u^{α}, λ' and their derivatives. For compressible materials, it may be shown (see for example, Adkins *et al.* 1954) that for a material which is unstressed in the undeformed state we may, to a sufficient degree of approximation, write

$$\frac{\partial W}{\partial J_1} = c_1 J_1, \quad \frac{\partial W}{\partial J_2} = c_2, \tag{12·6}$$

where c_1, c_2 are the values of $\partial^2 W/\partial J_1^2$, $\partial W/\partial J_2$ respectively at $J_1 = J_2 = J_3 = 0$. These constants are connected with the classical modulus of rigidity μ, and Poisson's ratio η of the elastic material by the relations

$$\mu = -2c_2, \quad \frac{c_1}{c_2} = -\frac{1-\eta}{1-2\eta}. \tag{12·7}$$

By introducing (12·1), (12·2), (12·5) and (12·6) into (6·10) we obtain

$$2c_1\lambda' + (c_1+c_2)\,\bar{a}' = 0. \tag{12·8}$$

Also, by combining (6·9) with (6·11) and making use of (12·1) to (12·8) in the resulting formula, we may obtain, for small deformations,

$$\left.\begin{aligned} \bar{n}_e^{11} &= -\frac{4h_0\mu}{1-\eta}\frac{\bar{a}_{12}\bar{a}_{22}}{\bar{a}^2}\bar{a}'_{12}, \quad \bar{n}_e^{22} = -\frac{4h_0\mu}{1-\eta}\frac{\bar{a}_{12}\bar{a}_{11}}{\bar{a}^2}\bar{a}'_{12}, \\ \bar{n}_e^{12} &= \frac{2h_0\mu}{1-\eta}\{2\bar{a}_{12}^2 + (1-\eta)\,\bar{a}\}\frac{\bar{a}'_{12}}{\bar{a}^2}, \end{aligned}\right\} \tag{12·9}$$

and the equation (7·1) for ϕ reduces to

$$\phi\,|_{12} = -\bar{a}\bar{n}_e^{12} = -\bar{a}\bar{n}^{12}, \tag{12·10}$$

where the single line denotes covariant differentiation with respect to the undeformed body, that is, with respect to the co-ordinates $\bar{\theta}^\alpha$ and the metric tensors $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$.

If the cords lie initially along the straight lines (11·2), the constraint conditions (12·4) become

$$u^1_{,1} - \beta u^2_{,1} = 0, \quad u^2_{,2} - \alpha u^1_{,2} = 0, \tag{12·11}$$

which have the solution

$$u^1 = \beta f_1(\bar{\theta}^1) + f_2(\bar{\theta}^2), \quad u^2 = f_1(\bar{\theta}^1) + \alpha f_2(\bar{\theta}^2), \tag{12·12}$$

where $f_1(\bar{\theta}^1)$, $f_2(\bar{\theta}^2)$ are arbitrary functions of their arguments. From (12·2), (11·3) and (12·9) we then have

$$\bar{a}'_{12} = f'_1 + f'_2, \tag{12·13}$$

$$\bar{n}_e^{11} = K_1(1+\alpha^2)\,(f'_1+f'_2), \quad \bar{n}_e^{22} = K_1(1+\beta^2)\,(f'_1+f'_2), \tag{12·14}$$

$$\bar{n}_e^{12} = K_2(f'_1+f'_2), \tag{12·15}$$

where

$$K_1 = \frac{4h_0\mu}{1-\eta}(\alpha+\beta), \quad K_2 = \frac{2h_0\mu}{1-\eta}\{2(\alpha+\beta)^2 + (1-\eta)\,(1-\alpha\beta)^2\}. \tag{12·16}$$

The Christoffel symbols formed with the metric tensors $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$ evidently vanish so that, from (12·10), (11·3) and (12·15) we obtain

$$\phi_{,12} = -K_2(f'_1+f'_2)/(1-\alpha\beta)^2,$$

or by integration

$$\phi = -K_2\{\bar{\theta}^1 f_2(\bar{\theta}^2) + \bar{\theta}^2 f_1(\bar{\theta}^1) + g_1(\bar{\theta}^1) + g_2(\bar{\theta}^2)\}/(1-\alpha\beta)^2, \tag{12·17}$$

where $g_1(\bar{\theta}^1)$, $g_2(\bar{\theta}^2)$ are further arbitrary functions. From (7·3) we obtain for the remaining stress resultant components

$$\bar{n}^{11} = -K_2(\bar{\theta}^1 f''_2 + g''_2), \quad \bar{n}^{22} = -K_2(\bar{\theta}^2 f''_1 + g''_1), \tag{12·18}$$

and expressions for the tensions in the cords may then be derived by combining these equations with (12·14), (7·9) and (5·15).

SHEET REINFORCED WITH A SINGLE SET OF CORDS

13. General formulation

If the elastic body contains a single set of cords, only one of the constraint conditions (3·10) applies. We shall assume that the cords coincide with the $\bar{\theta}^1$-curves so that $ds^2 = ds_0^2$ when $\bar{\theta}^2 =$ constant, or

$$\bar{a}_{11} = \bar{A}_{11}. \qquad (13\cdot1)$$

The methods and formulae of §§ 4 and 5 may again be employed, provided that the terms arising from the second set of cords are omitted throughout. The formulae of § 6 for the elastic terms in the stress resultants are evidently unchanged, and we may again introduce the Airy stress function ϕ to satisfy the equations of equilibrium. Equations (7·5) are now, however, replaced by

$$n^{\alpha\beta} = \bar{n}_e^{\alpha\beta} + \bar{\sigma}\frac{\partial\theta^\alpha}{\partial\bar{\theta}^1}\frac{\partial\theta^\beta}{\partial\bar{\theta}^1} = \epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\phi\,\|_{\gamma\rho} \quad (\bar{\sigma} = \bar{\sigma}_1), \qquad (13\cdot2)$$

or, if the moving co-ordinate system θ^α is chosen to coincide with $\bar{\theta}^\alpha$,

$$\bar{n}^{11} = \bar{n}_e^{11} + \bar{\sigma} = \phi\,\|_{22}/\bar{A}, \qquad (13\cdot3)$$

$$\bar{n}^{22} = \bar{n}_e^{22} = \phi\,\|_{11}/\bar{A}, \quad \bar{n}^{12} = \bar{n}_e^{12} = -\phi\,\|_{12}/\bar{A}. \qquad (13\cdot4)$$

From the results of § 6, it follows that for a given material, $\bar{n}_e^{\alpha\beta}$ are known functions of the metric tensors $\bar{A}_{\alpha\beta}$, $\bar{A}^{\alpha\beta}$, $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$ and (13·1) and (13·4) may therefore be regarded as three equations for the determination of the two displacement components and ϕ.

14. Small deformations

The linear equations for classically small deformations may be obtained by the method of § 12. In this case, some simplification of the formulae may be achieved by choosing the $\bar{\theta}^2$-curves to be the orthogonal trajectories of the paths followed by the cords in the undeformed body, so that

$$\bar{a}_{12} = 0, \quad \bar{a} = \bar{a}_{11}\bar{a}_{22}. \qquad (14\cdot1)$$

Equations (12·1), (12·2), (12·5) to (12·8) and the first of (12·3) continue to apply, but in place of (12·4) and the second of (12·3) we now have

$$\bar{a}'_{11} = 0 \quad \text{or} \quad x^\mu_{,1}u^\mu_{,1} = 0 \qquad (14\cdot2)$$

and

$$\bar{a}' = \bar{a}'_{22}/\bar{a}_{22}. \qquad (14\cdot3)$$

Also, (12·9) and (12·10) must be replaced by

$$\bar{n}_e^{11} = \frac{2h_0\mu\eta}{1-\eta}\frac{\bar{a}'_{22}}{\bar{a}}, \quad \bar{n}_e^{22} = \frac{2h_0\mu}{1-\eta}\frac{\bar{a}'_{22}}{\bar{a}_{22}^2}, \quad \bar{n}_e^{12} = 2h_0\mu\frac{\bar{a}'_{12}}{\bar{a}}, \qquad (14\cdot4)$$

and

$$\phi\,|_{11} = \bar{a}\bar{n}_e^{22} = \bar{a}\bar{n}^{22}, \quad \phi\,|_{12} = -\bar{a}\bar{n}_e^{12} = -\bar{a}\bar{n}^{12}, \qquad (14\cdot5)$$

respectively.

If the cords lie initially in parallel straight lines, we may, without loss of generality choose the co-ordinate systems $\bar{\theta}^\alpha$, x^α to coincide, so that

$$\left.\begin{aligned} \bar{\theta}^\alpha = x^\alpha, \quad x^\alpha_{,\beta} = \delta^\alpha_\beta, \\ \bar{a}_{\lambda\mu} = \bar{a}^{\lambda\mu} = \delta_{\lambda\mu}, \quad \bar{a} = 1. \end{aligned}\right\} \qquad (14\cdot6)$$

From the second of (12·2) we have

$$\bar{a}'_{\alpha\beta} = u^{\alpha}_{,\beta} + u^{\beta}_{,\alpha}, \tag{14·7}$$

and since the Christoffel symbols formed with the metric tensors $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$ now vanish, (14·4) and (14·5) yield

$$\phi_{,11} = \frac{4h_0\mu}{1-\eta} u^2_{,2}, \quad \phi_{,12} = -2h_0\mu(u^1_{,2}+u^2_{,1}). \tag{14·8}$$

From (14·7) and (14·2) we have

$$u^1_{,1} = 0 \quad \text{or} \quad u^1 = f(x^2), \tag{14·9}$$

where $f(x^2)$ is an arbitrary function of its argument. By eliminating ϕ from (14·8) and making use of (14·9) we obtain

$$(1-\eta)\, u^2_{,11} + 2u^2_{,22} = 0, \tag{14·10}$$

which may be solved to give

$$u^2 = \Omega'(\zeta) + \bar{\Omega}'(\bar{\zeta}), \tag{14·11}$$

where $\Omega'(\zeta)$ is an arbitrary function of

$$\zeta = x^1\sqrt{2} + \mathrm{i}x^2\sqrt{(1-\eta)}, \tag{14·12}$$

and $\bar{\Omega}'(\bar{\zeta})$ is its complex conjugate. By introducing (14·9) and (14·11) into (14·8) and integrating the resulting expressions we obtain

$$\phi = 2h_0\mu\left\{\frac{\mathrm{i}[\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})]}{\sqrt{(1-\eta)}} - x_1 f(x^2) + g(x^2)\right\}, \tag{14·13}$$

and hence

$$\begin{aligned}\bar{\sigma} &= \bar{n}^{11} - \bar{n}^{11}_e = \phi_{,22} - \bar{n}^{11}_e \\ &= -2h_0\mu\left\{\frac{\mathrm{i}(1+\eta)}{\sqrt{(1-\eta)}}[\Omega''(\zeta)-\bar{\Omega}''(\bar{\zeta})] + x^1 f''(x^2) - g''(x^2)\right\},\end{aligned} \tag{14·14}$$

$g(x^2)$ being a further arbitrary function of its argument.

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